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The traces of Hecke operators in the space of the 'Hilbert modular' type cusp forms of weight two.

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Introduction.

The purpose of the present note is to calculate the trace of Hecke operators acting in the space of the cusp forms of weight two belonging to a Hilbert modular group over a totally real algebraic number field. More generally, we carry it out for a discontinuous groups acting on $\mathfrak{Z}_n$, which consists of all $z=(z^{(1)},---,z^{(n)})$ with $z^{(i)}\in \mathbf{C}$, Im $z^{(i)}\neq 0$. Namely, let G be the product of n copies of $GL_2(\mathbf{R})$, considering of G as a group of transformations in $\mathfrak{Z}_n$. Let Γ be a subgroup of G operating on $\mathfrak{Z}_n$ discontinuously with a fundamental domain of finite volume. Let G^0 be the connected component of the identity of G, and set $\Gamma^0=\Gamma\cap G^0$. We denote by Z(G) the center of G and by ι the canonical homomorphism of G onto G/Z(G). It is assumed through out this paper that

(G.1) $\lambda(\Gamma^{o})$ is an irreducible subgroup of $\lambda(G^{o})$ such that

$\lambda(G^{o})/\lambda(\Gamma^{o})$ is non-compact and of finite measure,

(G.2) $\lambda(\Gamma^{o})$ satisfies the assumption (F) in [8] .

We fix once for all an element α in G^{o} such that Γ and $\alpha\Gamma\alpha^{-1}$ are commensurable, and denote by Γ' the subgroup of G generated by Γ and α. Let χ be a linear character of Γ'. We assume that χ satisfies

(C.1) the kernel of Γ_{χ} of χ in Γ is of finite index in Γ,

(C.2) $\chi(\varepsilon)=1$ for $\varepsilon \in Z(\Gamma)$ $(=\Gamma \cap Z(G))$.

Let k be an even integer. Let $T=T(\Gamma\alpha\Gamma)$ be the Hecke operator acting on the space of cusp forms of weight k with respect to Γ and χ; we denote above space by $S(\Gamma,k,\chi)$. We calculate the trace of T for the case k=2, $n>1$. For the case of $k>2$, the trace of T has been explicitly calculated in Shimizu [9]. Also for the case of k=2, the trace has been calculated in our previous papers [4], [5] under the condition of n=1 or the condition that Γ has a compact fundamental domain in $\mathfrak{H}_n$.

§1. A few facts from [5] .

Let H be the direct product of n complex upper half planes. Let $S(\Gamma^{o})$ be the set of all restrictions of the cusp forms in $S(\Gamma,2,\chi)$ to H. In this and next sections, from now on, we consider that T is restricted to $S(\Gamma^{o})$. Let us recall a few facts from [5]. We fix once for all a fundamental domain D of Γ^{o} in H. Let $\kappa_1,---,\kappa_h$ be all Γ^{o}_{χ}-inequivalent cusps belonging to D. g_p denotes an element of G^{o} such that $g_p\infty=\kappa_p$. Set $B=\Gamma^{o}\alpha\Gamma^{o}$, $B_p^{(1)}=\{\gamma\in B;\ \gamma\kappa_p=\kappa_p\}$, $\Gamma_p^{o(1)}=\{\gamma\in\Gamma^{o};\ \gamma\kappa_p=\kappa_p\}$ and $\Gamma_p^{o}=\{\gamma\in\Gamma_p^{o(1)};\ \gamma$ is a parabolic$\}$. Let $\tilde{H}=H\times(R/2\pi Z)^n$, $\tilde{D}=D\times(R/2\pi Z)^n$ with elements (z,ϕ) $(\phi=(\phi^{(1)},---,\phi^{(n)}))$ and we identify $\phi^{(i)}$ and $\phi^{(i)}+2\pi$. Let $\tilde{G}^{o}=G^{o}\times(R/2\pi Z)^n$ with elements (g,θ), and it acts on the space (z,ϕ) as

$$(g,\theta)(z,\phi) = (gz,(\phi^{(i)}+\arg(c^{(i)}z^{(i)}+d^{(i)})-\theta^{(i)})),\quad g^{(i)}=\begin{pmatrix} a^{(i)}, & b^{(i)} \\ c^{(i)}, & d^{(i)} \end{pmatrix}.$$

Let $L_o^2(\tilde{D})$ be the space of measurable functions $F(z,\phi)$ on $\tilde{H}$ taking values in C and satisfying the following conditions:

(i) $F(\gamma(z,\phi))=\chi(\gamma)\,F(z,\phi)$ for $\gamma\in\Gamma^{o}$,

(ii) $\int_{\tilde{D}}F(z,\phi)\,\overline{F(z,\phi)}\,dz\,d\phi<\infty$, $\left(dz=\prod_{i=1}^{n}\dfrac{dx^{(i)}dy^{(i)}}{y^{(i)2}},\ d\phi=\prod_{i=1}^{n}d\phi^{(i)}\right)$,

(iii) $\int_{R^n/M_p}F(g_p(z,\phi))\,dx^{(1)}---dx^{(n)}=0$ $(1\leq p\leq h)$, where $M_p = \{ \mu = (\mu^{(1)}, ---, \mu^{(n)});\ (g_p^{-1}\gamma g_p)^{(i)} = z^{(i)} + \mu^{(i)},\ \gamma \in \Gamma_p^o \}$. Let k_s be a $\tilde{G}^o$-invariant integral operator defined by a point pair invariant kernel: for $s > 0$,

$$(1.1)\quad k_s(z,\phi,z',\phi') = \prod_{i=1}^{n} \Big\{ (\exp(-2\sqrt{-1}(\phi^{(i)} - \phi'^{(i)}))) \times \left[\frac{(y^{(i)}y'^{(i)})^{\frac{1}{2}}}{(z^{(i)} - \bar{z}'^{(i)})/2\sqrt{-1}} \right]^2 \frac{(y^{(i)}y'^{(i)})^{s/2}}{|(z^{(i)} - \bar{z}'^{(i)})/2\sqrt{-1}|^s} - \frac{s}{2+s} \frac{(y^{(i)}y'^{(i)})^{1+s/2}}{|(z^{(i)} - z'^{(i)})/2\sqrt{-1}|^{2+s}} \Big\}.$$

It is well known that the ring of all $\tilde{G}^o$-invariant differential operators is generated by

$$(1.2)\quad \frac{\partial}{\partial \phi^{(i)}},\ \tilde{\Delta}^{(i)} = y^{(i)2}\left(\frac{\partial}{\partial x^{(i)2}} + \frac{\partial}{\partial y^{(i)2}} \right) + y^{(i)} \frac{\partial}{\partial x^{(i)}} \frac{\partial}{\partial \phi^{(i)}},\quad (1 \leq i \leq n).$$

Denote by $M(m,\lambda)$ the subspace of $L_0^2(\tilde{D})$ consisting of φ satisfying the following conditions

$$\frac{\partial}{\partial \phi^{(i)}} \varphi = -\sqrt{-1}m^{(i)}\varphi,\quad \tilde{\Delta}^{(i)}\varphi = \lambda^{(i)}\varphi \quad (1 \leq i \leq n).$$

By the general theory, the eigenvalues of k_s only depend on (m,λ); so we write the eigenvalue of k_s with $h_s(m,\lambda)$. The following proposition comes from [5, Proposition 1 & 2].

PROPOSITION. 1 The eigenspace $M(m,\lambda)$ in which k_s does not vanish and its eigenvalue are in the following table. The notations are defined as follows. In the series C, J is denoted a proper subset of $[1,n]$ and $I \cup J = [1,n]$; $\lambda_\rho^{(i)}$ ranges

Series	m	λ	Isomorphic to $M(m,\lambda)$	Eigenvalue $h_s(m,\lambda)$ of k_s	Trace of T
B	$m^{(i)} = 2$ $(1 \leq i \leq n)$	$\lambda^{(i)}=0$	$S(\Gamma^0)$	$\left(8\pi 2^s \dfrac{\Gamma(1+\frac{s}{2})^2\,\Gamma(\frac{1}{2})\Gamma(\frac{1+s}{2})}{\Gamma(1+s)\;\Gamma(2+\frac{s}{2})}\right)^n$	t_o
C	$m^{(i)}=0,2$ $m^{(j)}=2$ $(i \in I)$ $(j \in J)$	$\lambda^{(i)}=\lambda_\rho^{(i)}$ $\lambda^{(j)}=0$	$M(\{0,2\},\{\lambda_\rho^{(i)},0\})$	$\prod_{i \in I}\{(-8\pi 2^s c(s)/\Gamma(1+s))\; \Gamma(\frac{s}{2}+\delta_\rho^{(i)})\Gamma(\frac{s+2}{2}-\delta_\rho^{(i)}) \times \prod\left(-8\pi 2^s \dfrac{\Gamma(1+\frac{s}{2})^2\Gamma(\frac{1}{2})\Gamma(\frac{1+s}{2})}{\Gamma(1+s)\Gamma(2+\frac{s}{2})}\right)$	$t_{m,\lambda}$
D	$m^{(i)} = 0$ $(1 \leq i \leq n)$	$\lambda^{(i)}=0$	C	$\left(-8\pi 2^s \dfrac{\Gamma(1+\frac{s}{2})^2\Gamma(\frac{1}{2})\Gamma(\frac{1+s}{2})}{\Gamma(1+s)\Gamma(2+\frac{s}{2})}\right)^n$	t_1

over all eigenvalues of $\Delta^{(i)}=y^{(i)2}\left(\frac{\partial^2}{\partial x^{(i)2}}+\frac{\partial^2}{\partial y^{(i)2}}\right)$ satisfying $M(\{0,2\},\{\lambda_\rho^{(i)},0\}) \neq \{0\}$ expect $\lambda_\rho^{(i)} \neq 0$; $\lambda_\rho^{(i)}=\delta_\rho^{(i)}(\delta_\rho^{(i)}-1)$. The series D appears only if χ is trivial. $c(s)=\frac{s}{2}\,\frac{\Gamma(\frac{1}{2})\Gamma(\frac{1+s}{2})}{\Gamma(2+\frac{s}{2})}$.

We shall carry the action of T to $M(2,0)$ by the isomorphism in the series B and extend it to $L_0^2(\tilde{D})$. We can expess T restricted to $M(m,\lambda)$ by k_s in the following way;

$$T(\Gamma\alpha\Gamma) = h_s(m,\lambda)^{-1}\int_{\tilde{D}} K_s(z,\phi,z',\phi')\, F(z',\phi')\, dz'd\phi', \tag{1.3}$$

$$K_s(z,\phi,z',\phi')=\sum_{g \in \Gamma^0\alpha\Gamma^0} \chi(g)\; k_s(z,\phi,g(z',\phi')).$$

But for $s>0$, the kernel k_s is of (a)-(b) type in the sense of [7], therefore K_s is absolutely convergent and uniformly, if $(z,\phi),(z',\phi')$ are contained in some compact subregion of $\tilde{H}$. But as the fundamental domain $\tilde{D}$ is non-compact, the operator K_s is not, generally, completely continuous.

§ 2. An operator H_s.

2.1. Now we shall define a series M. Put $h_s(\delta)=h_s(2,\lambda)$ for simplicity. $e=(e^{(1)},---,e^{(n)})$ denotes a combination of $e^{(i)}=0$ or 2 $(1\leq i\leq n)$. For a complex number σ with $\mathrm{Re}(\sigma)>1$, we set

$$M_p^e(z,\phi,z',\phi';\sigma)=(2\pi)^{-n}\sum_{\{g\}\in\Gamma_p^o\backslash\Gamma^o}\chi(g)^{-1}\int_{\mathrm{Re}(\delta^{(1)})}---\int_{\mathrm{Re}(\delta^{(n)})}h_s(\delta)$$

$$(2.1)\qquad \times\prod_{i=1}^{n}\{\, b_p^{e^{(i)}}(g^{(i)}(z^{(i)},\phi^{(i)});\delta^{(i)}+\sigma-\tfrac{1}{2})\ \overline{b}_p^{e^{(i)}}(z'^{(i)},\phi'^{(i)};\delta^{(i)})\ d\delta^{(i)}\},$$

$$M'^{e}_p(z,\phi,z',\phi';\sigma)=\sum_{\nu}\chi(\beta_\nu)^{-1}M_p^e(\beta_\nu(z,\phi),z',\phi';\sigma),$$

where $b^{e^{(i)}}(z^{(i)},\phi^{(i)};u)=\exp(-\sqrt{-1}e^{(i)}(\phi^{(i)}+\arg(c_{g_p^{-1}}^{(i)}z^{(i)}+d_{g_p^{-1}}^{(i)})))(\mathrm{Im}\ g_p^{(i)-1}z^{(i)})^u$, and that $\Gamma^o a\Gamma^o=\bigcup_\nu\Gamma^o\beta_\nu$ (disjoint union).

For simplicity, we may assume that $\zeta_1=\infty, g_1=1$, $e=(0,---,0)$ in this and next paragrapg and treat M_1^0 mainly; we shall M, $\Gamma_\infty^{(1)}$, Γ_∞ instead of M_1, $\Gamma_1^{o(1)}$, Γ_1^o. By a simple calculation, we get

$$M(z,z';\sigma)=\sum_{\{g\}\in\Gamma_\infty^{(1)}\backslash\Gamma^o}\chi(g)^{-1}\prod_{i=1}^{n}(\mathrm{Im}\ gz)^{(i)\sigma}a(\mathrm{Im}\ gz,y'),$$

where $\quad a(y,y')=\sum\prod_{i=1}^{n}((\lambda yy'^{-1})^{\frac{1}{2}}+(\lambda yy'^{-1})^{-\frac{1}{2}})^{(i)-(1+s)}$,

and $\quad \Lambda_\infty=\{(\lambda^{(i)})=(a^{(i)}d^{(i)-1});\ g\in\Gamma_\infty^{(1)}\}$. It follows from [8, No.17] that $a(y,y')\leqq k$, k being a constant independent of y,y' and s. Thus, by the same way as in [8, Lemma 12], M converges absolutely and is holomorphic respect to σ for $\mathrm{Re}(\sigma)>1$. Further, from the definition, it follows immediately that

$M(gz,z';\sigma)=\chi(g)\,M(z,z';\sigma)$ for $g\in\Gamma^{o}$.

2.2. In this paragraph, we shall obtain the analytic continuation of M to the domain $\mathrm{Re}(\sigma)>\frac{1}{2}$, minus the interval $(\frac{1}{2},1]$, Now, we need some notations and propositions. We define Eisenstein series attached to the cusp κ_p by

$$E_p(z,\sigma)=\sum_{\{g\}\in\Gamma_p^{o(1)}\backslash\Gamma^{o}}\chi(g)^{-1}\prod_{i=1}^{n}(\mathrm{Im}\ g_p^{-1}z)^{(i)\sigma}. \tag{2.2}$$

By a simple calculation, the constant term of the Fourier expansion of $E_p(z,\sigma)$ is given

$$\delta_{pq}\prod_{i=1}^{n}y^{(i)\sigma}+\prod_{i=1}^{n}y^{(i)1-\sigma}\varphi_{pq}(\sigma), \tag{2.3}$$

where $\varphi_{pq}(\sigma)=\left(\frac{\Gamma(\sigma-\frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(\sigma)}\right)^{n}\sum_{\{g\}\in g_p^{-1}\Gamma_p^{o(1)}g_p\backslash g_p^{-1}(\Gamma^{o}-\Gamma_p^{o(1)})g_q/g_q^{-1}\Gamma_q^{o}g_q}\left(\prod^{n}(\det g^{(i)}/|c^{(i)}|^{2})\right)^{\sigma}\chi(g)^{-1}$,

and $\delta_{pq}=1$ or 0 according as $p=q$ or not.

PROPOSITION 2. $\varphi_{pq}(\sigma)$ may be continued holomorphically to the domain $\mathrm{Re}(\sigma)>\frac{1}{2}$, $\sigma\notin(\frac{1}{2},1]$.

This proof comes from [6, Theorem 3.1.1] with a little modification. Let $F(z,\sigma)$ be an analytic function of z,σ which is automorphic with respect to Γ^{o}, whose constant term of the Fourier expansion at κ_p $(1\leq p\leq h)$ has the form :

$$c_p(\sigma)\prod_{i=1}^{n}(\mathrm{Im}\ g_p^{-1}z)^{(i)\sigma}+d_p(\sigma)\prod_{i=1}^{n}(\mathrm{Im}\ g_p^{-1}z)^{(i)1-\sigma}.$$

For $Y>0$, we define the function $F^Y(z,\sigma)$ by

$$F^Y(z,\sigma)=\begin{cases} F(z,\sigma)-(c_p(\sigma)\prod_{i=1}^{n}(\mathrm{Im}\ g_p^{-1}z)^{(i)\sigma}+d_p(\sigma)\prod_{i=1}^{n}(\mathrm{Im}\ g_p^{-1}z)^{(i)1-\sigma}, & \text{if } \prod_{i=1}^{n}(\mathrm{Im}\ g_p^{-1}z)^{(i)}>Y \\ F(z,\sigma) & \text{otherwise.} \end{cases}$$

Then the Fourier expansion gives

LEMMA. If F ia a function as above, we have

$$(2.4)\quad d(\Lambda_p)^{-2}(E_p^Y(z,\sigma),F^Y(z,\sigma'))=\frac{\overline{c}_p(\sigma')Y^{\sigma+\overline{\sigma}'-1}}{\sigma+\overline{\sigma}'-1}+\frac{\overline{d}_p(\sigma')Y^{\sigma-\overline{\sigma}'}}{\sigma-\overline{\sigma}'}$$
$$-\sum_{q=1}^{h}\left(\frac{\varphi_{pq}(\sigma)\overline{d}_q(\sigma')Y^{-(\sigma+\overline{\sigma}'-1)}}{\sigma+\overline{\sigma}'-1}+\frac{\varphi_{pq}(\sigma)\overline{c}_q(\sigma')Y^{\overline{\sigma}'-\sigma}}{\sigma-\overline{\sigma}'}\right),$$

where $d(\Lambda_q)=\det(l_j^{(i)})$, $\lambda_1,---,\lambda_{n-1}$ being generators of Λ_q, and $l_j^{(i)}=\log\lambda_j^{(i)}$ $(1\leq j<n)$, $l_n^{(i)}=1/n$. Using above formula, we get the following proposition by same arguments as in [6, Theorem 3.2.2, 4.2.1.-4.2.3,& 4.3.1.-4.3.5].

PROPOSITION 3. $E_p(z,\sigma)$ is holomorphic in the domain $\mathrm{Re}(\sigma)>\frac{1}{2}$ expect at point of finite number which are simple poles of $\varphi_{pq}(\sigma)$ on $(\frac{1}{2},1]$, Moreover E_p and φ_{pq} have a unique and finite limit σ tending to a point on the line $\mathrm{Re}(\sigma)=\frac{1}{2}$.

Now we come back to $M(z,z';\sigma)$. The constant term of this Fourier expansion at ∞ is given by

$$(2.5)\quad \prod_{i=1}^{n}(y^{(i)})^{\sigma}\alpha(y,y')+\prod_{i=1}^{n}(y^{(i)})^{1-\sigma}\varphi_{11}(\sigma)\beta(y,y';\sigma),$$
$$\beta(y,y';\sigma)=\left(\frac{\Gamma(\sigma)}{\Gamma(\sigma-\frac{1}{2})\Gamma(\frac{1}{2})}\right)^n\int_{R^n}\prod_{i=1}^{n}\frac{du^{(i)}}{(u^{(i)2}+1)^{\sigma}}\alpha\left(\left(\frac{y^{(i)}}{(u^{(i)2}+1)}\right),y'\right).$$

Using the Fourier expansion of M, we get

PROPOSITION 4. $M(z,z';\sigma)$ can be continued holomorphically to the domain $\mathrm{Re}(\sigma)>\frac{1}{2}$ minus points which are poles of $\wp(\delta)$ belonging to $(\frac{1}{2},1]$. Moreover $M(z,z';\sigma)$ has a unique and finite limit for any sequence $\{\sigma_n\}$ of complex numbers such that $\mathrm{Re}(\sigma_n)>\frac{1}{2}$, $\lim \mathrm{Re}(\sigma_n)=\frac{1}{2}$.

2.3. Now we shall construct an operator H_s. Let $\{\mu_1,---,\mu_n\}$ be a basis of M_p and $d(M_p)=\det(\mu_j^{(i)})$. The kernel of H_s will be defined by

$$(2.6)\quad H_s(z,\phi,z',\phi')=\left(\frac{2^s c(s)}{2\pi}\right)^n \sum_{p=1}^{h} d(M_p)^{-1}\sum_{e}(-1)^{n-\Sigma\frac{e^{(i)}}{2}} \times M'^{e}_{p}(z,\phi,z',\phi';\tfrac{1}{2}),$$

where e runs over all conbination of $e^{(i)}=0$ or 2 $(1\leq i\leq n)$. By the direct calculation, when z and z' tend simultaneously towards the cusp κ_p, the kernel $H_s(z,\phi,z',\phi')$ is approximately equal to $\sum_{g\in B_p^{(1)}}\chi(g)\, k_s(z,\phi,g(z',\phi'))$. It follows that

$$K_s^*(z,\phi,z',\phi')=K_s(z,\phi,z',\phi')-H_s(z,\phi,z',\phi')$$

is bounded for all (z,ϕ), $(z',\phi')\in\tilde{H}$; therefore an integral operator K_s^* turns to be completely continuous. Moreover, by the same way as [4, §§ 4.3-4.4], we see that, for $F\in L_0^2(\tilde{D})$ which is an eigenfunction of $\frac{\partial}{\partial \phi^{(i)}}$ and $\tilde{\Delta}^{(i)}$, an eigenvalue of F for K_s^* is equal to that for K_s, and that the image of K_s^* is contained in $L_0^2(\tilde{D})$. Considering the trace K_s^* in $L_0^2(\tilde{D})$ with the same argument as [5, §3], we obtain

$$(2.7) \qquad t_0 = -(-1)^n t_1 + \lim_{s \to 0} \int_{\tilde{D}} K_s^*(z,\phi,z,\phi)\, dz\, d\phi.$$

Define the equivalence relation of elements of B by

$$(2.8) \qquad g \sim g' \iff g' = \varepsilon \gamma g \gamma^{-1} \text{ for } \gamma \in \Gamma^0,\ \varepsilon \in Z(\Gamma^0).$$

Let $[g]$ denote an equivalence class in B containing g. Let $\Gamma^0(g)$ be the group of all $\gamma \in \Gamma^0$ such that $\gamma g \gamma^{-1} = \varepsilon g$ for some $\varepsilon \in Z(\Gamma)$ and F_g (resp. F_g^* ; $\tilde{D}^*$) a fundamental domain of $\Gamma^0(g)$ in H (resp. $\Gamma^0(g)$ in H* ; Γ^0 in $\tilde{H}^*$) (H* being a subregion of H obtained by substracting the neibourhood of each parabolic point of Γ^0 from H, and $\tilde{H}^* = H^* \times (R/2\pi Z)^n$). We can rewrite

$$\mathrm{tr} \int_{\tilde{D}^*} K_s(z,\phi,z,\phi)\, dz\, d\phi = (2\pi)^n \sum_{[g],\, g \in B} \chi(g) \int_{F_g^*} k_s(z,0,z,0)\, dz.$$

For simplicity, we denote by $A(g,s;H^*)$ each term of the right hand side of above formula.

§3. An explicit formula for trace of $T(\Gamma\alpha\Gamma)$.

3.1. In this section, we shall calculate the trace of

$T(\Gamma\alpha\Gamma)$ in $S(\Gamma,2,\chi)$. Firstly, we classify an element in B. $g \in B$ is of one of the following types; (i) $g \in B \cap Z(G^{0})$, (ii) g is elliptic, (iii) g is hyperbolic and no fixed point of g is a parabolic point of Γ^{0}, (iv) g is hyperbolic and one of the fixed points of g is a parabolic point, (v) g is parabolic, (vi) g is mixed.

When g is of type (i),(ii),(iii) or (vi), $A(g,s,H^*)$ has been calculated in [5, § 4].

3.2. Case iv). We may assume that g leaves each of ∞ and 0 fixed. For $Y, Y' > 0$, put $F^*_g = \{ z = (r^{(i)}\exp(\sqrt{-1}\theta^{(i)}))$; $\log r^{(i)} = \sum u^{(i)} l^{(i)}_j$, for $0 < u_j < 1$ $(1 \leq j < n)$, $\log(Y'^{-1}\prod_{i=1}^{n}|\sin\theta^{(i)}|) < u_n < \log(Y\prod_{i=1}^{n}|\sin\theta^{i}|^{-1})$, $0 < \theta^{(i)} < \pi$ $(1 \leq i \leq n)\}$. Writing $g^{(i)}z^{(i)} = \lambda^{(i)}z^{(i)}$, and $\rho^{(i)} = \left|\frac{\lambda^{i}+1}{\lambda^{i}-1}\right|$, we have

$$A(g,s;H^*) = (-8\pi 2^{s})^{n}\chi(g)\,|\det(l^{(i)}_j)|\,\log(YY'\prod|\sin\theta^{(i)}|^{2})\prod_{i=1}^{n}|\lambda^{(i)}-1|^{(2+s)}$$
$$\times\int_0^{\pi}\cdots\int_0^{\pi}\prod_{i=1}^{n}\frac{|\tan\theta^{(i)}|^{s}\{1+\hat{c}(s)+(\hat{c}(s)-1)\rho^{(i)}\tan^{2}\theta^{(i)}}{(1+\rho^{(i)2}\tan^{2}\theta^{(i)})^{2+s/2}\cos^{2}\theta^{(i)}}\,d\theta^{(i)},$$

$(\hat{c}(s) = s/(2+s))$. Therefore, if $n > 1$, $A(g,s;H^*)$ is vanishes.

3.3. Case v). Consider the contribution of the parabolic classes in $\Gamma\alpha\Gamma$ on $\mathfrak{H}^n$. We may assume $\zeta_p = \infty$, $g_p = 1$. In this paragraph, let us use the notations in [9, §3.4]. For $Y > 0$, we put

$$F_g^* = \{ z=(x^{(i)}+\sqrt{-1}y^{(i)});\ x^{(i)}=\sum_{j=1}^{n} v_j \mu(\gamma_j^{(i)}) \text{ for } 0<v_j<1, 0<\prod_{i=1}^{n}|y^{(i)}|<Y\}.$$

Then we have

$$w=\sum_{g\in L_p}(2\pi)^n\chi(g)\int_{F_g^*}k_s(z,0,z,0)\,dz$$

$$=\lim_{\varepsilon\to 0}(-4\pi 2^{\varepsilon})^n\sum_{g\in L_p}\frac{d(g)\chi(g)}{m(g)^{1+\varepsilon}}\varepsilon^n\left(\frac{\Gamma(\frac{1+\varepsilon}{2})\Gamma(\frac{s-\varepsilon+1}{2})}{\Gamma(2+\frac{s}{2})}\right)^n+o(Y^{-1}).$$

By [9, Lemma 3.2], the series has at most a pole of order 1 at $\varepsilon=0$. By the assumption of $n>1$, it follows that $w=0$.

3.4. By a simple calculation, $\lim_{s\to 0}\operatorname{tr}\int_{\tilde{D}^*}H_s(z,\phi,z,\phi)\,dz\,d\phi=0$. Summing up the above results, we obtain

THEOREM 1. If $n>1$, the trace of $T(\Gamma a\Gamma)$ in $S(\Gamma,2,\chi)$ is given by the following formula :

$$\text{(3.1)}\qquad \operatorname{Tr} T(\Gamma a\Gamma)=\delta_1(4\pi)^{-n}v(\Gamma\backslash\mathfrak{H}_n)\chi(g_0)$$

$$+\sum_{\{g\}\in\mathfrak{S}}\frac{(-1)^n\chi(g)}{(\Gamma(g):Z(\Gamma))}-\delta_2(-1)^n\frac{2^n d}{(\Gamma:\Gamma^0)}.$$

The notations used in this formula are defined as follows :

$$\delta_1=\begin{cases}1 & \text{--- if } \Gamma a\Gamma\cap Z(G)\neq\emptyset\\ 0 & \text{--- otherwise}\end{cases},\qquad \delta_2=\begin{cases}1 & \text{--- if } \chi \text{ is trivial}\\ 0 & \text{--- otherwise}\end{cases},$$

$g_0\in\Gamma a\Gamma\cap Z(G)$,

$v(\Gamma\backslash\mathfrak{H}_n)$; the volume of a fundamental domain of Γ in $\mathfrak{H}_n$ relative to the invariant measure dz,

$\mathfrak{S}$; a complete system of inequivalent elliptic elements in $\Gamma a\Gamma$ with respect to the equivalence relation (2.8), d ; the number of right Γ-cosets in $\Gamma\alpha\Gamma$.

§4. The Hilbert modular groups.

Let Φ be a totally real algebraic number field of degree n over Q, and $A=M_2(\Phi)$. We denote by $\mathfrak{g}$, E_0, E_0^+, $\mathfrak{O}$, $\mathfrak{J}$ and U the ring of integers in Φ, the group of units in $\mathfrak{g}$, the subgroup of E_0 containing of all totally positive units, a maximal order in A, the idèle group of A and the idèle x such that $x_{\mathfrak{p}}$ is a unit of $\mathfrak{O}_{\mathfrak{p}}$ for all finite prime $\mathfrak{p}$, respectively. Writing $\Phi^{(i)}$ $(1\leq i\leq n)$ for the completion of Φ with respect to the infinite valuation of Φ and $A^{(i)}=A\otimes_{\Phi}\Phi^{(i)}$, every $x\in\mathfrak{J}$ is made to act on $\mathfrak{H}_n$ by $x(z)=(x^{(1)}(z^{(1)}),---,x^{(n)}(z^{(n)}))$ $(x^{(i)}\in A^{(i)})$. Then Γ satisfies our assumptions (G.1) and (G.2), and $\Gamma\backslash\mathfrak{H}_n$ is not compact. Let $\mathfrak{M}$ be an integral two-sided ideal in $\mathfrak{O}$ of norm $\mathfrak{a}$, and χ a linear character of $(\mathfrak{O}/\mathfrak{M})^*$; we consider χ as a character of $V_{\mathfrak{a}}$ $(=\{x\in\mathfrak{J}; x_{\mathfrak{p}}\in U_{\mathfrak{p}}$ for all $\mathfrak{p}|\mathfrak{a}\}$) by means of a natural homomorphism of $V_{\mathfrak{a}}$ onto $(\mathfrak{O}/\mathfrak{M})^*$. Then χ satisfies (C.1). We assume that χ satisfies (C.2). $\mathfrak{J}$ is a finite union of double cosets of U and A^* in the following way :

$\mathfrak{J} = \bigcup_{\lambda=1}^{h} Ux_\lambda A^*$ ($x_\lambda \in V_{\mathfrak{a}}$, h is the class number of $\mathfrak{O}$).

Put $U_\lambda = x_\lambda^{-1} U x_\lambda$ and $\Gamma_\lambda = A^* \cap U_\lambda$ ($1 \leq \lambda \leq h$). Let S_λ be the space of all cusp forms on $\mathfrak{H}_n$ of type $(\Gamma_\lambda, 2, \chi)$ and S the direct produ. t of $S_1, ---, S_h$. For an integral ideal $\mathfrak{q}$ in $\mathfrak{g}$, wedenote by $T(\mathfrak{q})$ the linear operator in S defined in [10, § 3.4]. Note that $T(\mathfrak{q}) \neq 0$ only if $\mathfrak{q}$ is a principal ideal and only if we can write $\mathfrak{q} = q\mathfrak{g}$ such that q is a totally positive element in $\mathfrak{g}$.

Combining Theorem 1 with [5, §5.1 & 8, §4], we obtain

THEOREM 2. Let $\mathfrak{q} = q\mathfrak{g}$ be a principal ideal in Φ with a totally positive element q. The trace of $T(\mathfrak{q})$ is given by

$$
\begin{aligned}
(4.1) \qquad \mathrm{Tr}.T(\mathfrak{q}) &= \delta(\mathfrak{q})\,(2\pi)^{-2n} 2h_0 D_0^{3/2} \zeta_0(2)\,\chi(q_0) \\
&\quad - \delta_2 (-1)^n h_0 (2^n/(E_0:E_0^+)) \sum_{\mathfrak{n}|\mathfrak{q}} N(\mathfrak{n}) \\
&\quad + (-1)^n \tfrac{1}{2} \sum_{\mathfrak{o}\in\Omega} \frac{h(\mathfrak{o})}{w(\mathfrak{o})} \sum_{a\in J(\mathfrak{o}),\, a \bmod E_0} \chi(a) \; .
\end{aligned}
$$

The notations are as follows. h_0, D_0 and ζ_0 are the class number of Φ, the discriminant of Φ over Q and the zeta function of Φ, respectively. $\delta(\mathfrak{q}) = 1$ if $\mathfrak{q} = q_0^2 \mathfrak{g}$ for some $q_0 \in \mathfrak{g}$ and otherwise $\delta(\mathfrak{q}) = 0$. $\delta_2 = 1$ if χ is a trivial and otherwise $\delta_2 = 0$. $\mathfrak{n}$ runs over all divisors of $\mathfrak{q}$. Ω is the set of all orders $\mathfrak{o}$ (taken up isomorphism) in totally imaginary quadratic extensions of Φ.

$h(\mathfrak{o})$ is the class number of $\mathfrak{o}$, and $w(\mathfrak{o})$ is the index of E_0 in the group of units in $\mathfrak{o}$. $J(\mathfrak{o})$ is the set of all $\alpha\in\mathfrak{o}$ such that $\alpha\notin\mathfrak{P}$, $N(\alpha)\mathfrak{J}=\mathfrak{q}$.

References.

(1) Eichler, M., Eine Verallgemeinung der Abelschen Integrale, Math. Z., 67 (1957), 267-298.

(2) Godement, R., The Spectral Decomposition of Cusp-Forms, Proc. Sympos. Pure Math., 9. Amer. Math. Soc., 1966, 225-233.

(3) Hirzebruch, F., The Hilbert modular group, resolution of the singularities at the cusp and related problems, Séminaire Bourbaki 1970/71, Exposé 396, 275-288.

(4) Ishikawa, H., On the trace formula for Hecke operators, J. Fac. Sci. Univ. Tokyo, Sec. I 20 (1973), 217-238.

(5) Ishikawa, H., On trace of Hecke operators for discontinuous groups operating on the product of the upper half planes, J. Fac. Sci. Univ. Tokyo, Sec. I 21 (1974),

(6) Kubota, T., Elementaly theory of Eisenstein series, Kodansha Scientific, 1973.

(7) Selberg, A., Harmonic analysis and discontinuous groups on weakly symmetric Riemann spaces with applications to

Dirichlet series, J. Indian Math. Soc. 20 (1956), 47-87.

(8) Shimizu, H., On discontinuous groups operating on the product of the upper half planes, Ann. of Math. 77 (1963), 33-71.

(9) Shimizu, H., On traces of Hecke operators, J. Fac. Sci. Univ. Tokyo, Sec. I 10 (1963), 1-19.

(10) Shimura, G., On Dirichlet series and abelian varieties attached to automorphic forms, Ann. of Math. 76 (1962), 237-294.

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