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# RESUME

## (A Note on Isolated Singularity)

Isao NARUKI

### 0. Notation

$(X,x)$: pair of analytic space and $x \in X$ such that $X\setminus x$ smooth.

$\iota$: $X\setminus x \hookrightarrow X$ (inclusion).

$\Omega_X^p$: sheaf of (analytic) p-forms on $X$ ($\mathcal{O}_X = \Omega_X^0$).

### 1. Duality

Lemma 1. $R^q\iota_*\iota^*\Omega_X^p \overset{\text{dual}}{\longleftrightarrow} R^{n-q-1}\iota_*\iota^*\Omega_X^{n-p}$ .

($1 \leq q \leq n-2$, $n=\dim X$).

Proof. Andreotti-Grauert [1].

### 2. Coherence of Local Cohomology

Lemma 2. $S$: coherent $\mathcal{O}_X$-Modele, $S|X\setminus x$: locally free

$\Rightarrow \mathcal{H}_x^q(S)$: coherent for $q < \dim X$

Proof. Siu [7].

### 3. Condition (L), Poincaré Lemma

Definition. $(X,x)$ satisfies $(L) \Leftrightarrow \mathcal{H}_x^q(\Omega_X^p)=0$ if $p+q<\dim X$.

Lemma 3. (Partial Poincaré Lemma). $(X,x)$ satisfies (L)

$\Rightarrow H^p(\Omega_{X,x}^{\cdot}) = 0$ $(0 < p < \dim X)$.

Proof. Bloom-Herrera [2].

## 4. Nice Function, Hypersurfacesection

Definition. $f: X \xrightarrow{\text{analytic}} \mathbb{C}$ is nice on $(X,x)$

$\Longleftrightarrow f(x)=0, \quad df_z \neq 0 \quad (\forall z \in X\backslash x).$

$(Y,y)$: hypersurfacesection of $(X,x)$

$\Longleftrightarrow \exists f$: nice on $(X,x)$ such that $(Y,y) \overset{\text{iso.}}{\simeq} (f^{-1}(0),x)$.

Lemma 4 (de Rham Lemma). $f$: nice on $(X,x)$, $(X,x)$ satisfies

(L) $\Longrightarrow 0 \to \Omega_X^0 \xrightarrow{df} \Omega_X^1 \to \cdots \to \Omega_X^{\dim X}$ exact.

## 5. Conservation of (L).

Let $(Y,y)$: hypersurface section of $(X,x)$

Theorem 1. $(X,x)$ satisfies (L)

$$\Longleftrightarrow \begin{cases} \text{i) } (Y,y) \text{ satisfies (L)} \\ \text{ii) } \dim \mathcal{H}_y^0(\Omega_Y^n) = \dim \mathcal{H}_y^1(\Omega_Y^n) \end{cases}$$

$(n = \dim Y \geq 2)$

Corollary. $(X,x)$: a complete intersection of hypersurfaces

$\Longrightarrow (X,x)$ satisfies (L)

Proof. Hamm [4].

## 6. Milnor Fibering

Let $f$: nice on $(X,x)$, $(Y,y)$: hypersurface section defined by $f$. We can assume by Milnor [6]

a) $(Y,y) \hookrightarrow (X,x) \xhookrightarrow{\text{closed}} (B,0)$ ($B$: open ball in $\mathbb{C}^N: (z_1, z_2, \cdots z_N)$)

b) $r|X\backslash x, r|Y\backslash y$ have no critical point $(r(z) = \sum_{i=1}^{N} |z_i|^2)$

Theorem 2. Under the above assumption, $\exists S$: a neighborhood of 0 in $\mathbb{C}$ such that i) $\mathbb{R}^p f_*(\Omega_f^{\cdot})|S$ are coherent $\mathcal{O}_S$-Modules

ii) $H^p(\Omega_{f,x}^{\cdot}) \cong \mathbb{R}^p f_*(\Omega_f^{\cdot})_0$

where we have set $\Omega_f^p = \Omega_X^p / df \wedge \Omega_X^{p-1}$ $(p=0,1,2,\cdots)$.

Proof. Brieskorn [3] and Kiehl-Verdier [5].

## 7. Main Theorem

Theorem 3. $(X,x)$, $f$, $(Y,y)$, $\Omega_f^{\cdot}$ as in §6, $(X,x)$ satisfies (L)

$\Rightarrow$

i) $H^p(\Omega_{f,x}^{\cdot}) = 0$ $\quad 1 \leq p \leq n-1$ $\quad (n = \dim Y)$.

ii) $H^n(\Omega_{f,x}^{\cdot})$: free $\mathcal{O}_{\mathbb{C},0}$-module of finite rank.

iii) $\mu = \operatorname{rank}_{\mathcal{O}_{\mathbb{C},0}} H^n(\Omega_{f,x}^{\cdot}) = \dim_{\mathbb{C}} H^n({}'\Omega_{Y,y}^{\cdot})$

$({}'\Omega_Y^{\cdot} = 0 \to \Omega_Y^0 \xrightarrow{d} \Omega_Y^1 \xrightarrow{d} \cdots \xrightarrow{d} \Omega_Y^n \longrightarrow 0\ )$.

iv) $\mu = \dim R^1 \iota_* \iota^* \Omega_Y^{n-1} + \dim H^n(\iota_* \iota^* \Omega_Y^{\cdot}) - \dim H^{n-1}(\iota_* \iota^* \Omega_p^{\cdot})$

v) In case $(X,x)$ is smooth, there are isomorphisms i

$$\mathcal{H}_y^0(\Omega_Y^{n+1}) \overset{i}{\simeq} \mathcal{H}_y^1(\Omega_Y^n) \overset{i}{\simeq} \cdots\cdots \overset{i}{\simeq} \mathcal{H}_y^{n-1}(\Omega_Y^2)$$

$$\mathcal{H}_y^1(\Omega_Y^{n-1}) \overset{i}{\simeq} \mathcal{H}_y^2(\Omega_Y^{n-2}) \overset{i}{\simeq} \cdots\cdots \overset{i}{\simeq} \mathcal{H}_y^{n-1}(\Omega_Y^1).$$

## Reference


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