数理解析研究所講究録
第 237 巻 1975 年 104-112

A 4-MANIFOLD WHICH ADMITS NO SPINES

By Yukio MATSUMOTO [1]

1. Introduction. In this note, we shall sketch the proof of the following result:

THEOREM 1. *There exists a compact 4-dimensional PL manifold W^4 with boundary satisfying the following conditions:*

(i) W^4 *is homotopically equivalent to the 2-torus* $T^2 = S^1 \times S^1$, *and*

(ii) *no homotopy equivalence* $T^2 \to W^4$ *is homotopic to a PL embedding*.

By a PL embedding is meant a one which is not necessarily locally flat. Theorem 1 is an application of the codimension two surgery theory developed in [4], [5], [6].

A calculation in the proof leads to another consequence concerned with submanifolds in codimension two. Let K^{4n} denote a product $\mathbb{C}P_2 \times \cdots \times \mathbb{C}P_2$ of n-copies of $\mathbb{C}P_2$.

THEOREM 2. *For each* $n \geq 0$, *there exists a locally flat embedding* $h_{(4n)}$ *of* $K^{4n} \times S^1$ *into the interior of* $K^{4n} \times D^2 \times S^1$, *which is homotopic to the zero cross section* $K^{4n} \times \{0\} \times S^1$, *but is not locally flatly concordant to a splitted embedding*.

A *splitted embedding* (with respect to a point $*$ of S^1) means a locally flat embedding $f : K^{4n} \times S^1 \to K^{4n} \times D^2 \times S^1$ such that (i) f is transverse regular to $K^{4n} \times \mathring{D}^2 \times \{*\}$, thus the

[1] The author is partially supported by the Fūjukai Foundation.

intersection $M^{4n} = f(K^{4n} \times S^1) \cap K^{4n} \times D^2 \times \{*\}$ is a closed manifold, and (ii) the inclusion $M^{4n} \longrightarrow K^{4n} \times D^2 \times \{*\}$ is a homotopy equivalence.

Theorem 2 contrasts with Farrell-Hsiang's result [2] which may be considered as the splitting theorem in higher codimensions.

Let $P_m(\pi \longrightarrow \pi')$ [2] be the group of Seifert forms introduced in [6]. Theorem 2 is equivalent to saying that Shaneson's formula on $L_m(\pi \times \mathbb{Z})$ [10] is not immediately generalized to a formula on $P_m((\pi \longrightarrow \pi') \times \mathbb{Z})$. See remarks after Lemmas 3 and 4.

Cappell and Shaneson [1] developed another method of surgery in codimension two from homology surgery point of view. They introduced groups $\Gamma_m(\pi \longrightarrow \pi')$ of singular Hermitian forms. Partial explanations about the relationship between Γ- and P-functors will be found in [7].

2. Construction of W^4. Let $h : S^1 \longrightarrow S^1 \times D^2$ be an embedding indicated in Fig. 1. Essentially the same embedding $S^1 \longrightarrow S^1 \times S^2$ was used by Mazur [8] to construct a contractible 4-manifold.

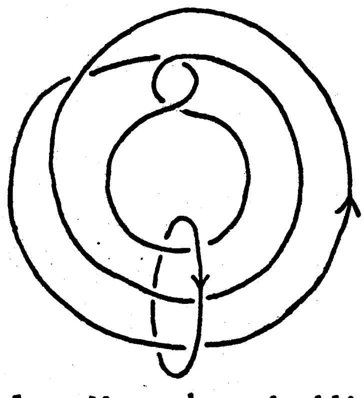

Fig. 1. Mazur's embedding

[2] This notation slightly differs from the original one used in [6].

Extend h to a framed embedding $\bar{h} : S^1 \times D^2 \to S^1 \times D^2$ in such a way that $\bar{h}$ followed by the natural inclusion $S^1 \times D^2 \to S^3$ is a trivial knot with a trivial framing. Let $\bar{g} : S^1 \times D^2 \to S^1 \times D^2$ be a thickened zero-section defined by, say, $\bar{g}(x, \xi) = (x, \frac{1}{2}\xi)$ for $(x, \xi) \in S^1 \times D^2$.

Then our manifold W^4 is constructed by taking a disjoint union $(S^1 \times D^2 \times I)_0 \cup (S^1 \times D^2 \times I)_1$ of 2-copies of $S^1 \times D^2 \times I$ and identifying $((x, \xi) \times \{1\})_0$ with $(\bar{h}(x, \xi) \times \{0\})_1$, and $((x, \xi) \times \{0\})_0$ with $(\bar{g}(x, \xi) \times \{1\})_1$. Since h is homotopic to $g = \bar{g} \mid S^1 \times \{0\}$, W^4 is homotopically equivalent to T^2.

3. Seifert forms. First, we give some definitions. Let $\pi \to \pi'$ be an onto homomorphism of groups whose kernel is generated by a (specified) central element t. A $(-1)^n$-Seifert form (over $\pi \to \pi'$) is a $(-1)^n t$-Hermitian form defined over $\mathbb{Z}\pi$ which is non-singular over $\mathbb{Z}\pi'$. We denote by $P_{2n}(\pi \to \pi')$ the 'Witt group' of $(-1)^n$-Seifert forms over $\pi \to \pi'$. For more precise definitions, see [6] or [7].

The geometric motivation is as follows. (In what follows, all manifolds are compact and oriented. All submanifolds are locally flat.) Suppose a pair (V^{2n+2}, M^{2n-1}) consisting of a connected 2n+2-manifold V^{2n+2} and a closed (possibly empty) 2n-1-submanifold M^{2n-1} of the boundary ∂V has the same simple homotopy type as a Poincaré pair (X, Y) of formal dimension $2n \geq 4$. One can find an exterior n-connected submanifold L^{2n} of V^{2n+2}

such that $\partial L^{2n} = M^{2n-1}$ [4]. Let N be a 2-disk bundle neighbourhood of L^{2n}.

The homomorphism $\pi_1(V - L) \longrightarrow \pi_1(V)$ is independent of the choice of a particular exterior n-connected submanifold L^{2n}. It is denoted by $\pi \longrightarrow \pi'$ and is said to be <u>associated</u> with (V, M). $\pi \longrightarrow \pi'$ has the property stated at the beginning of this section (t being represented by the fiber of the associated S^1-bundle with N.)

The codimension two intersection form [6] defines a $(-1)^n$-Seifert form (λ, μ) (over $\pi \longrightarrow \pi'$) on the left $\mathbb{Z}\pi$-module $\pi_{n-1}(V-L, N-L)$.

Moreover, the element of $P_{2n}(\pi \longrightarrow \pi')$ which the form represents does not depend on L^{2n}, but depends only on (V, M). Denote the element by $\eta(V, M)$. Then it is proven that V admits a locally flat spine cobounding M if and only if $\eta(V, M) = 0$, provided that $2n \geqq 6$ [6].

Now we will return to our present situation. With the notations of § 2, we denote the disjoint union $h(S^1 \times \{0\}) \times \{0\} \cup -g(S^1 \times \{0\}) \times \{1\}$, which is a submanifold of $\partial(S^1 \times D^2 \times I)$, by Σ^1. Denote the pair $(S^1 \times D^2 \times I, \Sigma^1)$ by Θ. Then $\Theta \times \mathbb{C}P_2$ is homotopically equivalent to $(S^1 \times I \times \mathbb{C}P_2, S^1 \times \{0, 1\} \times \mathbb{C}P_2)$, and the homomorphism associated with it is $\mathbb{Z} \times \mathbb{Z} \longrightarrow \mathbb{Z}$ $(= (\mathbb{Z} \to 1) \times \mathbb{Z})$.

LEMMA 1. <u>The element</u> $\eta(\Theta \times \mathbb{C}P_2)$ <u>of</u> $P_6((\mathbb{Z} \longrightarrow 1) \times \mathbb{Z})$ <u>is represented by the</u> (-1)-<u>Seifert form</u> (G, λ, μ) <u>given by</u>:

$G = \Lambda x_1 \oplus \Lambda x_2$, $\lambda(x_1, x_2) = -s^{-1}$, $\mu(x_1) = s - 1$, $\mu(x_2) = -1$, where $\Lambda = \mathbb{Z}[t, t^{-1}, s, s^{-1}]$, t (or s) denoting the (positive) generator of the first (or the second) $\mathbb{Z}$ of $(\mathbb{Z} \to 1) \times \mathbb{Z}$.

The proof of Lemma 1 is divided into three steps. The first is to construct a 2-surface F^2 of genus 1 in $S^1 \times D^2 \times I$ cobounding Σ^1. To F^2 are attached two 2-disks D_1, D_2 within $S^1 \times D^2 \times I$. To compute the codimension two intersection [6] of these specific 2-disks is the second and crucial step which requires careful geometric observations. The final one is to lift this low dimensional computation to the higher dimensional one by crossing (with) $\mathbb{C}P_2$. Cf. [6, pp.307-308].

Remark. The matrix $(\lambda(x_i, x_j))$ of the Seifert form of Lemma 1 is

$$\begin{pmatrix} (s-1)-t(s^{-1}-1), & -s^{-1} \\ st, & -1+t \end{pmatrix}.$$

The determinant of this matrix is $s(t-1) + (t^2-t+1) + s^{-1}(t-t^2)$, which coincides (up to units) with the Alexander polynomial of Mazur's link (Fig. 1) calculated by the method of Torres-Fox [11].

4. The Murasugi invariant. Let $(\Lambda x_1 \oplus \cdots \oplus \Lambda x_{2\ell}, \lambda, \mu)$ be a (-1)-Seifert form over $(\mathbb{Z} \to 1) \times \mathbb{Z}$, Λ denoting $\mathbb{Z}[t, t^{-1}, s, s^{-1}]$. The Murasugi invariant σ_M of the form is defined to be the signature of the symmetric integral matrix obtained from $(\lambda(x_i, x_j))$ by substituting $t = s = -1$.

It gives us a well defined homomorphism

$$\sigma_M : P_{4k+2}((\mathbb{Z} \to 1) \times \mathbb{Z}) \to \mathbb{Z}.$$

Let (G, λ, μ) be the form given in Lemma 1. We have

$$\sigma_M(G, \lambda, \mu) = \operatorname{sign}\begin{pmatrix} -4 & 1 \\ 1 & -2 \end{pmatrix} = -2.$$

This implies

LEMMA 2. $\eta(\Theta \times \mathbb{C}P_2)$ <u>is a non-zero element of</u> $P_6((\mathbb{Z} \to 1) \times \mathbb{Z})$.

<u>Remark</u>. It should be noted that the value -2 is the minus of Murasugi's ξ-invariant [9] of Mazur's link (Fig. 1).

Again let $(\Lambda x_1 \oplus \cdots \oplus \Lambda x_{2\ell}, \lambda, \mu)$ be a (-1)-Seifert form over $(\mathbb{Z} \to 1) \times \mathbb{Z}$. Then the substitution $s = 1$ gives us a (-1)-Seifert form over $\mathbb{Z} \to 1$, defining a homomorphism ρ_+ : $P_{4k+2}((\mathbb{Z} \to 1) \times \mathbb{Z}) \to P_{4k+2}(\mathbb{Z} \to 1)$. ρ_+ is a left inverse of the 'inclusion' homomorphism $i_* : P_{4k+2}(\mathbb{Z} \to 1) \to P_{4k+2}((\mathbb{Z} \to 1) \times \mathbb{Z})$.

Now $\rho_+(\eta(\Theta \times \mathbb{C}P_2))$ is represented by the form (G', λ', μ') given by $G' = \Lambda' x_1 \oplus \Lambda' x_2$, $\lambda'(x_1, x_2) = -1$, $\mu'(x_1) = 0$, $\mu'(x_2) = -1$, where $\Lambda' = \mathbb{Z}[t, t^{-1}]$. This form is null-cobordant in the sense of [6, § 4.9]. (The submodule $\Lambda' x_1$ is a Seifert subkernel.) Therefore, $\rho_+(\eta(\Theta \times \mathbb{C}P_2)) = 0$. This together with Lemma 2 yields

LEMMA 3. $\eta(\Theta \times \mathbb{C}P_2)$ <u>is not in the image of</u> $i_* : P_6(\mathbb{Z} \to 1) \to P_6((\mathbb{Z} \to 1) \times \mathbb{Z})$.

<u>Remark</u>. The cokernel of i_* is proven <u>not</u> to be finitely generated.

5. <u>Proofs of theorems</u>.

<u>Proof of Theorem 1</u>. Let W^4 be the manifold constructed in §2. The manifold $W^4 \times \mathbb{C}P_2$ is homotopically equivalent to

$T^2 \times \mathbb{C}P_2$, and the homomorphism $(\mathbb{Z} \to 1) \times \mathbb{Z} \times \mathbb{Z}$ is associated with it (§ 3). The element $\eta(W^4 \times \mathbb{C}P_2)$ is proven to be the image of $\eta(\Theta \times \mathbb{C}P_2)$ under the (injective) homomorphism j_* : $P_6((\mathbb{Z} \to 1) \times \mathbb{Z}) \to P_6((\mathbb{Z} \to 1) \times \mathbb{Z} \times \mathbb{Z})$.

Now suppose that there were a spine $T_0^2 \subset W^4$. T_0^2 may be assumed to be locally flat except at one point. The product $T_0^2 \times \mathbb{C}P_2$ is a spine of $W^4 \times \mathbb{C}P_2$ with the singularity of the type (knot cone) $\times \mathbb{C}P_2$. Since $\pi_1(\{pt\} \times \mathbb{C}P_2) = \{1\}$, this singularity is replaced by a (7, 5)-knot cone singularity [4]. This means that $\eta(W^4 \times \mathbb{C}P_2)$ $(= j_*(\eta(\Theta \times \mathbb{C}P_2)))$ is in the image of $j_* \circ i_*$, since C_5, the knot cobordism group of (7,5)-knots, is isomorphic to $P_6(\mathbb{Z} \to 1)$ [6]. However, this contradicts Lemma 3.

Proof of Theorem 2. Let M^m be a closed 1-connected manifold of dimension $m \geq 5$, $f : M^m \times S^1 \to M^m \times D^2 \times S^1$ a locally flat embedding which is a homotopy equivalence. Denote the pair $(M^m \times D^2 \times S^1 \times I,\ f(M^m \times S^1) \times \{0\} \cup M^m \times \{0\} \times S^1 \times \{1\})$ by Ψ. The homomorphism $(\mathbb{Z} \to 1) \times \mathbb{Z}$ is associated with Ψ.

LEMMA 4. (i) *If* m *is odd*, f *is splittable. In other words*, f *is locally flatly concordant to a splitted embedding*. (ii) *If* m *is even*, f *is splittable if and only if* $\eta(\Psi)$ *is in the image of* $P_{m+2}(\mathbb{Z} \to 1) \to P_{m+2}((\mathbb{Z} \to 1) \times \mathbb{Z})$.

Let $h_{(4n)} : K^{4n} \times S^1 \to K^{4n} \times D^2 \times S^1$ be defined by $h_{(4n)} = id_K \times h$, h being Mazur's embedding. Then Theorem 2 follows from Lemmas 3 and 4.

Remark. Lemma 4 is generalized to non-simply connected manifolds as follows: There is no obstruction in the odd dimensional case. In the even dimensional case, the obstruction lies in the cokernel of $P_{m+2}(\pi \longrightarrow \pi') \oplus L_{m+1}(\pi') \longrightarrow P_{m+2}((\pi \longrightarrow \pi') \times \mathbb{Z})$, but even in the latter case any embedding is almost splittable in the sense of [3].

6. Concluding remarks. 1) For each $g \geq 1$, one can construct a spineless 4-manifold of the same homotopy type as the orientable surface of genus g.

2) If we start the construction with the embedding indicated in Fig. 2, we will obtain $W^{4'}$ which admits a locally flat spine.

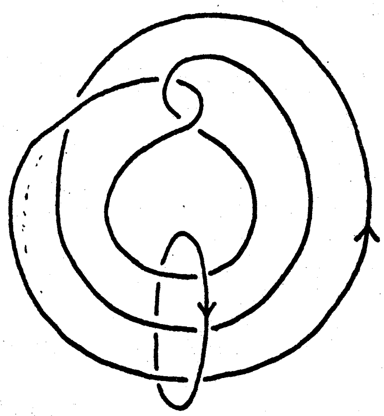

Fig.2 False embedding

REFERENCES

1. S. E. Cappell and J. L. Shaneson, The codimension two placement problem and homology equivalent manifolds, Ann. of Math. 99 (1974), 277-348.

2. F. T. Farrell and W. C. Hsiang, A geometric interpretation of the Künneth formula for algebraic K-theory, Bull. Amer. Math. Soc. 74 (1968), 548-553.

3. ———— , Manifolds with $\pi_1 = \mathbb{Z} \times_\alpha G$, to appear.

4. M. Kato and Y. Matsumoto, Simply connected surgery of submanifolds in codimension two I, J. Math. Soc. Japan 24 (1972), 586-608.

5. Y. Matsumoto, Surgery and singularities in codimension two, Proc. Japan Acad. 47 (1971), 153-156.

6. ———— , Knot cobordism groups and surgery in codimension two, J. Fac. Sci. Univ. Tokyo, Sec. IA 20 (1973), 253-317.

7. ———— , Some relative notions in the theory of Hermitian forms, Proc. Japan Acad. 49 (1973), 583-587.

8. B. Mazur, A note on some contractible 4-manifolds, Ann. of Math. 73 (1961), 221-228.

9. K. Murasugi, On the signature of links, Topology 9 (1970), 283-298.

10. J. L. Shaneson, Wall's surgery obstruction groups for $\mathbb{Z} \times G$, Ann. of Math. 90 (1969), 296-334.

11. G. Torres and R. H. Fox, Dual presentations of the group of a knot, Ann. of Math. 59 (1954), 211-218.

Department of Mathematics
College of General Education
University of Tokyo
Komaba, Meguro-ku
Tokyo, 153
Japan