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An example of surfaces with $b_1 = 1$

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In this note, we consider (compact complex) surfaces S satisfying

(α) $b_1(S) = 1$, $b_2(S) \neq 0$,

and

(β) S is minimal.

(cf. [1] and [3].) As yet, we have two kinds of such surfaces. We shall construct one of them. (Another kind of them is constructed in [4].) In our construction, we shall use some methods in Hirzebruch [2].

For a quadratic irrational number x, we denote by x' the conjugate of x. We take a real quadratic irrational number ω such that

$$\omega > 1 > \omega' > 0.$$

Then ω is expanded into a purely periodic continued fraction:

$$\omega = [[n_0, n_1, \cdots, n_{r-1}]]$$

where r is the smallest period (see [2]). For any integer i, we define

$$n_i = n_j, \quad 0 \leqq j \leqq r-1, \quad i \equiv j \pmod{r}.$$

Then

$$n_i \geqq 2 \quad \text{for any integer} \quad i.$$

We define quadratic irrational numbers ω_i inductively by

$$(1) \qquad \begin{aligned} \omega_0 &= \omega \\ \omega_{i-1} &= n_{i-1} - 1/\omega_i . \end{aligned}$$

Let $\mathfrak{M}$ be the $\mathbb{Z}$-module generated by 1 and ω, and let

$$U = \{\alpha \in \mathbb{R} \mid \alpha > 0, \quad \alpha \cdot \mathfrak{M} = \mathfrak{M}\},$$

$$U^+ = \{\alpha \in U \mid \alpha \cdot \alpha' = 1\}.$$

Then U and U^+ are infinite cyclic groups. We take a generator α_1 of U^+ such that $\alpha_1 > 1$. Throughout this note we assume that <u>there exists an element</u> α_0 <u>of</u> U <u>such that</u>

$$\alpha_0 \cdot \alpha_0' = -1, \quad \alpha_0 > 1.$$

Then α_0 generates U and

$$\alpha_1 = \alpha_0^2 .$$

We define integral matrices N_0 and N_1 by

$$(2) \qquad {}^tN_i \cdot \begin{bmatrix}\omega \\ 1\end{bmatrix} = \alpha_i \begin{bmatrix}\omega \\ 1\end{bmatrix}, \quad i = 0,1.$$

Then

$$\det N_0 = -1, \quad N_0^2 = N_1,$$

$$N_1 = \begin{bmatrix} n_{r-1}, & 1 \\ -1, & 0 \end{bmatrix} \cdot \begin{bmatrix} n_{r-2}, & 1 \\ -1, & 0 \end{bmatrix} \cdot \cdots \cdot \begin{bmatrix} n_0, & 1 \\ -1, & 0 \end{bmatrix}$$

For example, if we take

$$\omega = (3 + \sqrt{5})/2,$$

then

$$\omega = [[3]],$$

$$\alpha_1 = (3 + \sqrt{5})/2, \quad \alpha_0 = (1 + \sqrt{5})/2,$$

$$N_1 = \begin{bmatrix} 3, & 1 \\ -1, & 0 \end{bmatrix}, \quad N_0 = \begin{bmatrix} 2, & 1 \\ -1, & -1 \end{bmatrix}$$

We define positive quadratic numbers a_i for any integer i as follows :

$$a_i = (\omega_1 \omega_2 \cdots\cdots \omega_i)^{-1} \quad \text{for} \quad i \geqslant 1,$$

$$a_0 = 1,$$

$$a_{-i} = \omega_0 \omega_{-1} \omega_{-2} \cdots\cdots \omega_{-i+1} \quad \text{for} \quad i \geqslant 1.$$

It follows from (1) that

$$\text{(3)} \qquad \begin{aligned} n_{i-1}\, a_{i-1} - a_i &= a_{i-2}, \\ n_{i-1}\, a'_{i-1} - a'_i &= a'_{i-2}. \end{aligned}$$

We take two series of infinitely many copies of $\mathbb{C}^2$:

$$V_i = \{(u_i, v_i) \in \mathbb{C}^2\}, \quad i \in \mathbb{Z},$$

$$W_j = \{(z_j, w_j) \in \mathbb{C}^2\}, \quad j \in \mathbb{Z}.$$

We identify (u_i, v_i) of V_i and (u_{i-1}, v_{i-1}) of V_{i-1} if and only if

$$v_i = v_{i-1}^{n_{i-1}}\, u_{i-1},$$

$$u_i = 1/v_{i-1}, \quad u_i \neq 0, \; v_{i-1} \neq 0,$$

and form their union

$$\mathcal{V} = \bigcup_{i \in \mathbb{Z}} V_i \; .$$

Similarly we form the union of W_j

$$\mathcal{W} = \bigcup_{j \in \mathbb{Z}} W_j$$

by the identifications

$$w_j = w_{j-1}^{n_{j-1}}\, z_{j-1},$$

$$z_j = 1/w_{j-1}, \quad z_j \neq 0, \; w_{j-1} \neq 0.$$

Then $\mathcal{V}$ and $\mathcal{W}$ are Hausdorff spaces and, hence, complex manifolds with $\{V_i\}$ and $\{W_j\}$ as their coordinate neighbourhoods, respectively. Let $\tilde{C}$ be a subvariety of $\mathcal{V}$ defined by

$$\tilde{C} \cap V_i = \{(u_i, v_i) \mid u_i \cdot v_i = 0\}$$

for any $i \in \mathbb{Z}$. Then $\tilde{C}$ consists of infinitely many irreducible components $\tilde{C}_i$, $i \in \mathbb{Z}$, where $\tilde{C}_i$ is a non-singular rational curve and

$$\tilde{C}_i \cap \tilde{C}_{i+1} = \text{the origin } p_{i+1} \text{ of } V_{i+1}, \quad \text{(transversally)}$$

$$\tilde{C}_i \cap \tilde{C}_k = \phi, \quad i - k \neq \pm 1, 0,$$

$$\tilde{C}_i^2 = - n_i.$$

Similarly, $\mathcal{W}$ contains a subvariety $\tilde{D}$ with infinitely many irreducible components $\tilde{D}_j$, $j \in \mathbb{Z}$, where $\tilde{D}_j$ is a non-singular rational curve and

$$\tilde{D}_j \cap \tilde{D}_{j+1} = \text{the origin } q_{j+1} \text{ of } W_{j+1}, \quad \text{(transversally)}$$

$$\tilde{D}_j \cap \tilde{D}_\ell = \phi, \quad j - \ell = \pm 1, 0,$$

$$\tilde{D}_j^2 = - n_j.$$

It is clear that

$$\mathcal{V} - \tilde{C} = \{(u_0, v_0) \in V_0 \mid u_0 \cdot v_0 \neq 0\},$$

$$\mathcal{W} - \tilde{D} = \{(z_0, w_0) \in W_0 \mid z_0 \cdot w_0 \neq 0\}.$$

By (3), we can prove that

$$|u_i|^{a_i} \cdot |v_i|^{a_{i-1}} = |u_k|^{a_k} \cdot |v_k|^{a_{k-1}},$$
$$|u_i|^{a'_i} \cdot |v_i|^{a'_{i-1}} = |u_k|^{a'_k} \cdot |v_k|^{a'_{k-1}}, \quad \text{on} \quad V_i \cap V_k,$$

and

$$|z_j|^{a_j} \cdot |w_j|^{a_{j-1}} = |z_\ell|^{a_\ell} \cdot |w_\ell|^{a_{\ell-1}},$$
$$|z_j|^{a'_j} \cdot |w_j|^{a'_{j-1}} = |z_\ell|^{a'_\ell} \cdot |w_\ell|^{a'_{\ell-1}}, \quad \text{on} \quad W_j \cap W_\ell.$$

Hence, if we define

$$r = |u_i|^{a_i} \cdot |v_i|^{a_{i-1}},$$
$$s = |u_i|^{a'_i} \cdot |v_i|^{a'_{i-1}}, \quad \text{on} \quad V_i,$$

and

$$p = |z_j|^{a_j} \cdot |w_j|^{a_{j-1}},$$
$$q = |z_j|^{a'_j} \cdot |w_j|^{a'_{j-1}}, \quad \text{on} \quad W_j,$$

then r, s and p, q are non-negative continuous functions on $\mathcal{V}$ and $\mathcal{W}$, respectively. Moreover

$$\tilde{C} = \{P \in \mathcal{V} \mid r(P) = 0\} = \{P \in \mathcal{V} \mid s(P) = 0\},$$
$$\tilde{D} = \{Q \in \mathcal{W} \mid p(Q) = 0\} = \{Q \in \mathcal{W} \mid q(Q) = 0\}.$$

We identify $(u_0, v_0) \in \mathcal{V} - \tilde{C}$ and $(z_0, w_0) \in \mathcal{W} - \tilde{D}$ if and only if

$$v_0 = w_0^a \cdot z_0^b,$$
$$u_0 = w_0^c \cdot z_0^d, \quad u_0, v_0, w_0, z_0 \neq 0,$$

where $\begin{bmatrix} a, & b \\ c, & d \end{bmatrix} = N_0$, and form the union of $\mathcal{V}$ and $\mathcal{W}$.

From (2) and (3) we can deduce

$$
(4) \qquad
\begin{aligned}
r(w_0^c\, z_0^d,\ w_0^a\, z_0^b) &= p(z_0,\ w_0)^{\alpha_0}, \\
s(w_0^c\, z_0^d,\ w_0^a\, z_0^b) &= q(z_0,\ w_0)^{-1/\alpha_0},
\end{aligned}
$$

for $(z_0, w_0) \in \mathcal{W} - \tilde{D}$. By (4), we can prove that $\mathcal{V} \cup \mathcal{W}$ is a Hausdorff space and, hence, a complex manifold. We define

$$
\rho(P) = \begin{cases} r(P) & \text{for } P \in \mathcal{V}, \\ p(P)^{\alpha_0} & \text{for } P \in \mathcal{W}, \end{cases}
$$

and

$$
\sigma(P) = \begin{cases} s(P) & \text{for } P \in \mathcal{V}, \\ q(P)^{-1/\alpha_0} & \text{for } P \in \mathcal{W}. \end{cases}
$$

The formula (4) implies that ρ and σ are <u>continuous</u> mappings of $\mathcal{V} \cup \mathcal{W}$ into $[0, \infty]$ and $[0, \infty]$, respectively. Clearly we obtain

$$
\begin{aligned}
\tilde{C} \cup \tilde{D} &= \{P \in \mathcal{V} \cup \mathcal{W} \mid \rho(P) = 0\}, \\
\tilde{C} &= \{P \in \mathcal{V} \cup \mathcal{W} \mid \sigma(P) = 0\}, \\
\tilde{D} &= \{P \in \mathcal{V} \cup \mathcal{W} \mid \sigma(P) = \infty\}.
\end{aligned}
$$

We define an analytic automorphism g of $\mathcal{V} \cup \mathcal{W}$ as follows :

g sends (u_i, v_i) of V_i to (u_i, v_i) of V_{i-r},

g sends (z_j, w_j) of W_j to (z_j, w_j) of W_{j-r}.

From (2), it follows that

$$
(5) \qquad
\begin{aligned}
g^*\rho &= \rho^{\alpha_1}, \\
g^*\sigma &= \sigma^{1/\alpha_1}.
\end{aligned}
$$

Let $\mathcal{D}$ be an <u>open</u> subset of $\mathcal{V} \cup \mathcal{W}$ defined as follows :

$$\mathfrak{D} = \{P \in \mathfrak{V} \cup \mathfrak{W} \mid \rho(P) < 1\}.$$

Then $\mathfrak{D}$ is invariant by g and

$$\mathfrak{D} \supset \tilde{C}, \tilde{D} .$$

Moreover

$$g(\tilde{C}_i) = \tilde{C}_{i-r}, \quad g(p_i) = p_{i-r},$$

$$g(\tilde{D}_j) = \tilde{D}_{j-r}, \quad g(q_j) = q_{j-r}.$$

By (5), we can prove that g generates a properly discontinuous group $\langle g \rangle$ of analytic automorphisms of $\mathfrak{D}$ free from fixed points. We define S_ω to be the quotient space of $\mathfrak{D}$ by $\langle g \rangle$:

$$S_\omega = \mathfrak{D}/\langle g \rangle .$$

S_ω is a complex manifold of dimension 2.

Let π be the projection of $\mathfrak{D}$ onto S_ω and let

$$C = \pi(\tilde{C}), \quad C_i = \pi(\tilde{C}_i),$$

$$D = \pi(\tilde{D}), \quad D_j = \pi(\tilde{D}_j).$$

Then C and D are compact subvarieties of S_ω which have irreducible components $C_0, C_1, \cdots, C_{r-1}$ and $D_0, D_1, \cdots, D_{r-1}$, respectively. When $r \geqslant 2$, C and D are cycles of non-singular rational curves $C_0, C_1, \cdots, C_{r-1}$ and $D_0, D_1, \cdots, D_{r-1}$, respectively, where the intersections are transversal and

$$C_i^2 = D_i^2 = -n_i.$$

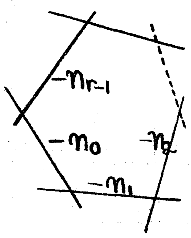
When $r = 1$, C and D are rational curves with an ordinary double point and

$$C^2 = D^2 = -n_0 + 2.$$

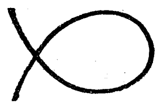
In any case

(6) $$C^2 = D^2 = -(n_0 + n_1 + \cdots + n_{r-1} - 2r).$$

The configurations for C, D are illustrated as follows :



$r \geqq 2$,



$r = 1$.

Evidently, C and D do not intersect.

Let (ξ, ζ) be the coordinate of $\mathbb{H} \times \mathbb{C}$ where $\mathbb{H}$ is the upper half of the complex plane and let G be the group of analytic automorphisms of $\mathbb{H} \times \mathbb{C}$ generated by

$$g_0 : (\xi, \zeta) \longrightarrow (\alpha_1 \xi, \frac{1}{\alpha_1}\zeta),$$

$$g_1 : (\xi, \zeta) \longrightarrow (\xi + \omega, \zeta + \omega'),$$

$$g_2 : (\xi, \zeta) \longrightarrow (\xi + 1, \zeta + 1).$$

G is a properly discontinuous group of automorphisms of $\mathbb{H} \times \mathbb{C}$ free from fixed points. We define a holomorphic mapping $\phi = (\xi, \zeta)$ of $S_\omega - C - D$ into $\mathbb{H} \times \mathbb{C} / G$ by

$$2\pi\sqrt{-1}\, \xi = \omega \log v_0 + \log u_0,$$

$$2\pi\sqrt{-1}\, \zeta = \omega' \log v_0 + \log u_0.$$

Then ϕ is an isomorphism of $S_\omega - C - D$ onto $\mathbb{H} \times \mathbb{C} / G$. By this fact and the fact that C, D are compact, we can prove that S_ω is a compact complex surface.

Since $\mathcal{D}$ is simply connected, we obtain

Proposition 1.

$$\pi_1(S_\omega) \cong \mathbb{Z}, \quad b_1(S_\omega) = 1.$$

Let K be the canonical line bundle of S_ω. Then

$$K = [-C - D].$$

Hence, by (6) and the Noether formula, we obtain

Proposition 2.

$$b_2(S_\omega) = c_2(S_\omega) = 2(n_0 + n_1 + \cdots + n_{r-1} - 2r).$$

By the adjunction formula, we can easily prove

Proposition 3. S_ω contains no irreducible curves other than $C_0, C_1, \cdots, C_{r-1}, D_0, D_1, \cdots, D_{r-1}$. In particular, S_ω is minimal. The above propositions imply that S_ω satisfies (α) and (β).

Remark.

1. If $\omega = (3 + \sqrt{5})/2$, then $b_2(S_\omega) = 2$ and S_ω contains exactly two irreducible curves.

2. Let $S^+ = \{P \in S_\omega | \sigma(P) \leq 1\}$ and $S^- = \{P \in S_\omega | \sigma(P) \geq 1\}$. Then S^+ and S^- are deformation retracts of C and D, respectively. Hence the Euler numbers of S^+ and S^- equal r. Moreover, the Euler number of $S^+ \cap S^-$ equals zero. Thus, by the additivity of the Euler number, we obtain

$$b_2(S_\omega) = c_2(S_\omega) = 2r,$$

$$n_0 + n_1 + \cdots + n_{r-1} = 3r.$$

3. S_ω has an involution ι defined as follows :

ι sends (u_i, v_i) of V_i to (u_i, v_i) of W_i,

ι sends (z_j, w_j) of W_j to (z_j, w_j) of V_j.

ι has no fixed points on S_ω and

$$\iota(C_i) = D_i.$$

Let

$$\hat{S}_\omega = S_\omega/\langle\iota\rangle.$$

$\hat{S}_\omega$ also satisfies (α) and (β).

4. F. Hirzebruch remarked that we can construct a surface for any real quadratic irrational number ω such that $\omega > 1 > \omega' > 0$ by similar methods.

5. D. Mumford, I. Nakamura and T. Oda remarked that S_ω can be constructed by the methods of troidal embeddings.

References

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