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An embedding theorem for polarized manifolds

By Takao Fujita

Department of Mathematics, University of Tokyo

0. Introduction

In this note, using the theory of Δ-genus, we establish an embedding theorem for polarized manifolds.

A pair (V,L) of a variety V and a line bundle L on V will be called a prepolarized variety. If L is ample, we call the pair (V,L) a polarized variety. Let $\chi(V,tL)=\sum_{p=0}^{n}(-1)^p\dim H^p(V,tL)=\sum_{j=0}^{n}\chi_j(V,L)t^{[j]}/j!$, $n=\dim V$, $t^{[j]}=\prod_{\alpha=0}^{j-1}(t+\alpha)$, be the expansion of the Hilbert polynomial. Then we define $d(V,L)=\chi_n(V,L)$, $g(V,L)=1-\chi_{n-1}(V,L)$ and $\Delta(V,L)=n+d(V,L)-\dim H^0(V,L)$. We remark that $d(V,L)=L^n=(c_1(L))^n\{V\}$ and that $2g(V,L)-2=(K_V+(n-1)L)L^{n-1}$ for a smooth V, K_V being its canonical bundle. Now we formulate our main result:

Theorem. Let (M,L) be a polarized manifold, i.e., a smooth polarized variety. Suppose that |L| has at most a finite number of base points. Then L is very ample if $d(M,L)\geqq 2\Delta(M,L)+1$ and if $g(M,L)\geqq\Delta(M,L)$.

In what follows we shall prove a little more general result.

Notation, Convention and Terminology

We employ the same notation as in [1], which is analogous to that of EGA. An analytic space is assumed to be compact, and a sheaf on it is assumed to be coherent. A variety is an irreducible reduced complex analytic space. A smooth variety is called a manifold. A vector bundle is not distinguished from the locally free sheaf associated with it.

By a divisor is meant a Cartier divisor.

$h^p(S,F):=\dim H^p(S,F)$, where F is a sheaf on an analytic space S.

$\chi(S,F):=\sum_{p=0}^{n}(-1)^p h^p(S,F)$, where $n=\dim S$.

$\{Z\}$: The homology class associated with an analytic cycle Z.

$[\Lambda]$: The line bundle associated with a linear system Λ of divisors.

$Bs\Lambda$: The set of base points of Λ.

ρ_Λ: The rational mapping associated with Λ, which turns out to be a morphism if $Bs\Lambda=\emptyset$.

$|L|$: The complete linear system of divisors associated with a line bundle L on S.

L_T: The pull back of L to a space T by a given morphism $T\to S$.

K_M: The canonical line bundle of a manifold M.

We use additive notation for tensor products of line bundles, and use multiplicative notation for cup products of their Chern classes.

An abbreviated form of this notation will be used when there is no danger of confusion.

1. The case of curves

In this section C is a curve, i.e., a variety of dimension 1.

Remark 1.1. A curve is locally Macaulay, i.e., $(O_C)_x$ is a Macaulay local ring for every $x\in C$.

Definition 1.2. A sheaf F on C is said to be quasi-invertible if it is invertible on an open dense subset of C and if it has no subsheaf whose support is a point.

Remark 1.3. i) Any invertible sheaf on C is quasi-invertible. ii) The tensor product of an invertible sheaf and a quasi-invertible sheaf is quasi-invertible, too.

iii) Any subsheaf of a quasi-invertible sheaf is quasi-invertible.

iv) The canonical sheaf ω of C is quasi-invertible.

Definition 1.4. For a quasi-invertible sheaf F on C we define degF to be $\chi(C,F)-1+g$, where $g=g(C)=h^1(C,O_C)$. $\Delta(C,F)$ is defined to be $1+\deg F-h^0(C,F)$. Needless to say, this definition coincides with the usual one when F is invertible.

Proposition 1.5. Let F,G be quasi-invertible sheaves and let $\varphi:F\to G$ be a non-trivial homomorphism. Then φ is injective, $\deg F\leq \deg G$ and $h^1(F)\geq h^1(G)$. Moreover, if degF=degG, φ is an isomorphism.

Proof. $\mathrm{Supp}(\varphi(F))$ is not a finite set since $0\neq\varphi(F)\subset G$. Hence φ is bijective at the generic point of C. So $\dim\mathrm{Supp}(\mathrm{Ker}(\varphi))\leq 0$ and this implies $\mathrm{Ker}(\varphi)=0$. The remaining assertion is clear since $\mathrm{SuppCoker}(\varphi)$ is a finite set.

Corollary 1.6. If $h^0(C,F)>0$ or if $\deg F\geq 0$, then $\Delta(C,F)\geq 0$.

Proposition 1.7. Let F be a quasi-invertible sheaf on C such that $\deg F\geq 2\Delta(C,F)+1\geq -1$. Then $\Delta(C,F)=g(C)$.

Proof. Note that $g(C)-\Delta(C,F)=h^1(C,F)$. Assume $g>\Delta$. Then there is an effective divisor D of degree $(\Delta+1)$ such that $h^0(C,\omega[-D])=g-\Delta-1$. We have $\mathrm{Hom}(O[D],F)\neq 0$ since $h^0(F[-D])\geq h^0(F)-\deg D=1+\deg F-\Delta-(\Delta+1)\geq 1$. Thus we obtain $h^1(F)\leq h^1([D])=h^0(\omega[-D])=g-\Delta-1$. This contradicts $h^1(F)=g-\Delta$.

Proposition 1.8. Let F be a quasi-invertible sheaf on C such that $\deg F\geq 2\Delta(C,F)\geq 0$. Then F is generated by its global sections.

Proof. Let G be the subsheaf of F generated by the global sections. Suppose $G\neq F$. Then $\deg G<\deg F$ and $\Delta(G)<\Delta(F)$ since $\deg F-\Delta(F)$

$=\deg G-\Delta(G)$. This also yields $\deg G > 2\Delta(G)$. Hence $g(C)=\Delta(G)<\Delta(F)$. This contradicts $\Delta(F)=g-h^1(F)$.

Theorem 1.9. Let F be a quasi-invertible sheaf on C with $\deg F \geqq 2g(C)$ and let L be an invertible sheaf with $\deg L \geqq 2g(C)+1$. Then the canonical homomorphism: $H^0(C,F)\otimes H^0(C,L)\to H^0(C,F\otimes L)$ is surjective.

Mumford [5] proved this result for a smooth curve C. After a slight modification his method also works in our general case.

2. Higher dimensional cases

Definition 2.1. Let (V,L) be a prepolarized variety. A member of $|L|$ is called a highest rung if it is irreducible and reduced. A sequence of subvarieties $V=V_n\supset V_{n-1}\supset\cdots\supset V_1$ of V is called a ladder if V_{j-1} is a highest rung of (V_j,L_{V_j}) for each $j=2,\cdots,n$.

Proposition 2.2. Let D be a highest rung of a prepolarized variety (V,L). Then $\chi_r(D,L_D)=\chi_{r+1}(V,L)$ for $r\geqq 0$. In particular, $d(D,L_D)=d(V,L)$ and $g(D,L_D)=g(V,L)$.

By definition, this follows from the exact sequence of sheaves $0\to O_V((t-1)L)\to O_V(tL)\to O_D(tL)\to 0$.

Proposition 2.3. Let V,L,D be as above. Then $\Delta(D,L_D)\leqq\Delta(V,L)$. The equality holds if and only if $|L|_D=|L_D|$, or equivalently, if and only if the restriction $r:H^0(V,L)\to H^0(D,L_D)$ is surjective.

This is clear since $\Delta(V,L)-\Delta(D,L)=\dim\mathrm{Coker}(r)$.

Definition 2.4. A line bundle L on a variety V is said to be fully generating if the canonical homomorphism $m_t:H^0(V,tL)\otimes H^0(V,L)\to H^0(V,(t+1)L)$ is surjective for every $t\geqq 1$.

Proposition 2.5. Let D be a highest rung of (V,L). Suppose that $|L_D|=|L|_D$ and that L_D is fully generating. Then

a) $r_t : H^0(V,tL) \to H^0(D,tL_D)$ is surjective for any $t \geq 1$,

b) L itself is fully generating.

Proof. We use the following commutative diagram:

$$\begin{array}{ccccc} & & H^0(V,kL)\otimes H^0(V,L) & \xrightarrow{r_k \otimes r_1} & H^0(D,kL)\otimes H^0(D,L) \\ & \nearrow & \downarrow m_k & & \downarrow m'_k \\ H^0(V,kL) & \longrightarrow & H^0(V,(k+1)L) & \xrightarrow{r_{k+1}} & H^0(D,(k+1)L) \end{array}$$

The dotted arrow is defined by $\varphi \mapsto \varphi\otimes\delta$, where $\delta \in H^0(V,L)$ is the defining section of D. Assuming a) for $t \leq k$, we see that $m'_k \circ (r_k \otimes r_1) = r_{k+1} \circ m_k$ is surjective. This implies that r_{k+1} is surjective. Moreover, combined with the exactness of the lower row, this yields the surjectivity of m_k, too.

Proposition 2.6. Let (V,L) be a prepolarized variety with $g(V,L) \geq \Delta(V,L)$. Suppose that (V,L) has a ladder with D being the highest rung of it.

a) If $d(V,L) \geq 2\Delta(V,L)-1$, then $|L|_D = |L_D|$ except dimV=1.

b) If $d(V,L) \geq 2\Delta(V,L)$, then $Bs|L| = \emptyset$.

c) If $d(V,L) \geq 2\Delta(V,L)+1$, then i) $g(V,L) = \Delta(V,L)$ and ii) L is fully generating.

Proof. We use induction on n=dimV since the result is proved for n=1 in the preceding section. When $n \geq 2$, we can apply the induction hypothesis to (D,L) since $\Delta(D,L) \leq \Delta(V,L) \leq g(V,L) = g(D,L)$. Assume that a) is false. Then $\Delta(D,L) < \Delta(V,L)$ and consequently $d(D,L) = d(V,L) \geq 2\Delta(D,L)+1$. It follows that $g(D,L) = \Delta(V,L)$. This contradicts $g(D,L) = g(V,L) \geq \Delta(V,L)$. Thus a) is proved. Now we infer that ~~g(V,L)~~ $Bs|L| = Bs|L|_D = Bs|L_D| = \emptyset$ if $d \geq 2\Delta$ and that $g(V,L) = g(D,L) = \Delta(D,L) = \Delta(V,L)$ if $d \geq 2\Delta+1$. Moreover, c-ii) follows from Proposition 2.5.

To establish a sufficient condition so that (V,L) has a ladder, we use the following result of Hironaka.

Theorem 2.7. Let Λ be a linear system on a variety V. Then there is a smooth variety V' with a birational morphism $\pi: V' \to V$ and a linear system Λ' on V' with $Bs\Lambda' = \emptyset$ such that $\pi^*\Lambda = F + \Lambda'$ with F being the maximal fixed component of $\pi^*\Lambda$.

We remark that $W = \rho_{\Lambda'}(V')$ is independent of the choice of (V', Λ')

Definition 2.8. Λ is said to be degenerate if $\dim W < \dim V$.

Proposition 2.9. Let (V,L) be a prepolarized variety such that V is locally Macaulay, $\dim Bs|L| \leqq 0$ and that $|L|$ is not degenerate. Then (V,L) has a ladder.

Proof. Let $\pi: V' \to V$, Λ', F be as in Theorem 2.7 so that $\pi^*|L| = F + \Lambda'$. Then a general member S of Λ' is non-singular and connected (see [4], Theorem 2). Letting D be the member of $|L|$ corresponding to S, we infer that $D = \pi(S)$ is irreducible and generically non-singular. Hence D is reduced since it is locally Macaulay. Since $\dim Bs|L_D| \leqq 0$ and $|L_D|$ is not degenerate, we can repeat such processes to obtain a ladder of (V,L).

Proposition 2.10. Let (M,L) be a prepolarized manifold such that $\dim Bs|L| \leqq 0$ and $d(M,L) \geqq 2\Delta(M,L) - 1$. If $|L|$ is degenerate, then a general member of $|L|$ is non-singular.

Proof. It suffices to show that for each $p \in Bs|L|$ there is a member D of $|L|$ which is non-singular at p. If otherwise, letting $q: M_1 \to M$ be the monoidal transform of M with center p, we infer that the fixed part of $q^*|L|$ is mE_p with $m \geqq 2$, where $E_p = q^{-1}(p)$. Let $\Lambda_1 = q^*|L| - mE_p$ and let M', $\pi: M' \to M_1$, Λ' and F be as in Theorem 2.7 so that $\pi^*\Lambda_1=F+\Lambda'$. Then we have $\dim W<n=\dim V$ and $[\Lambda']$ is the pull back of the hyperplane bundle H on P^N. $LH^{n-1}\{M'\}=L(L-mE_p-F)^{n-1}=L^n>0$ since $Bs|L|$ is a finite set. Therefore $\dim W=n-1$. Letting $w=\deg W$ we have $0\leq\Delta(W,H)\leq(n-1)+w-h^0(M,L)=w-d+\Delta-1$. On the other hand, letting $L_1=[\Lambda_1]$, we can easily show that $L^n-m^n=L_1{}^n\geq L_1H^{n-1}=(L-mE_p)H^{n-1}=L^n-mE_pH^{n-1}$. Hence $E_pH^{n-1}=wE_p\{X\}>0$ ~~and therefore $E_pX\geq 1$~~ for a general fiber X of $\rho_{\Lambda'}$ and therefore $E_p\{X\}\geq 1$. Now we have $d=d(M,L)=L^n=LH^{n-1}=(H+F+mE_p)H^{n-1}\geq mE_pH^{n-1}\geq mw\geq 2w\geq 2(d-\Delta+1)$. This contradicts the assumption $d\geq 2\Delta-1$.

Corollary 2.11. A prepolarized manifold (M,L) has a ladder if $d(M,L)\geq 2\Delta(M,L)-1$, $d(M,L)>0$ and $\dim Bs|L|\leq 0$.

Proof. In view of Proposition2.9 we may assume that $|L|$ is degenerate. Then a general member D of $|L|$ is non-singular. As is well-known $Bs|mL|=\emptyset$ for a large integer m, because $Bs|L|$ is a finite set. Then $|mL|$ is not degenerate since $d(M,L)>0$. Hence we have $h^1(M,-L)=0$ (see [4]). Therefore D is connected and becomes a highest rung. It is clear that we can repeat such processes.

Theorem 2.12. Let (M,L) be a polarized manifold such that $g(M,L)\geq\Delta(M,L)$ and $\dim Bs|L|\leq 0$. Then

a) $Bs|L|=\emptyset$ if $d(M,L)\geq 2\Delta(M,L)$,

b) L is very ample and fully generating if $d(M,L)\geq 2\Delta(M,L)+1$.

This follows from Proposition 2.6 since (M,L) has a ladder.

3. Applications

Theorem 3.1. Let (V,L) be a polarized variety. Then $\dim Bs|L|<\Delta(V,L)$.

For a proof, see Fujita[1].

Now we show a simple application of Theorem 2.12. Of course there are many other applications which will be published elsewhere.

Theorem 3.2. Let (M,L) be a polarized manifold such that $d(M,L)=3$ and $h^0(M,L)=n+2$, $n=\dim M$. Then M is a hypercubic in P^{n+1}.

Proof. Since $\Delta(M,L)=1$, we have $\dim Bs|L| \leqq 0$. Hence it suffices to prove the following lemma in order to apply Theorem 2.12.

Lemma 3.3. Let (M,L) be a polarized manifold such that $g(M,L)=0$ and (M,L) has a ladder. Then $\Delta(M,L)=0$.

Proof. Let $M=V_n \supset V_{n-1} \supset \cdots \supset V_1$ be a ladder. We claim that $h^p(V_j,-tL)=0$ for $0 \leqq p \leqq j$, $1 \leqq t \leqq j-1$. To prove this we use induction on j from above. Suppose $j=n$. Then the claim follows from the vanishing theorem of Kodaira for $p<n$. As for $p=n$, we have $h^n(M,-tL)=h^0(M,K_M+tL)=0$ for $t \leqq n-1$ since $(K_M+(n-1)L)L^{n-1}=2g(M,L)-2<0$. Suppose that the claim is true for $j \geqq k+1$. Then $h^p(V_{k+1},-tL)=h^{p+1}(V_{k+1},-(t+1)L)=0$ for $0 \leqq p \leqq k$, $1 \leqq t \leqq k-1$. Hence $h^p(V_k,-tL)=0$ since there is an exact sequence $H^p(V_{k+1},-tL) \to H^p(V_k,-tL) \to H^{p+1}(V_{k+1},-(t+1)L)$. Thus the claim is proved and in particular $h^1(V_j,-L)=0$. Therefore we infer $h^1(V_n) \leqq h^1(V_{n-1}) \leqq \cdots \leqq h^1(V_1)=g(V_1)=g(M,L)=0$, and consequently $\Delta(M,L)=\Delta(V_{n-1},L)=\cdots=\Delta(V_1,L)=0$ since $V_1 \simeq P^1$.

Similarly we can prove the following

Theorem 3.4. Let (M,L) be a polarized manifold with $d(M,L)=4$ and $\Delta(M,L)=1$. Then M is a complete intersection of type $(2,2)$.

References

[1] Fujita, T.; On the structure of polarized varieties of Δ-genus zero, (to appear) .

[2] Hironaka, H.; Resolution of singularities of an algebraic variety over a field of characteristic zero I-II, Ann. Math., 79(1964), 109-326.

[3] Kodaira, K.; On a differential-geometrical method in the theory of analytic stacks, Proc.Nat.Acad.Sci.U.S.A.,39(1953),1268.

[4] Mumford, D.; Pathologies III, Amer. J. Math.,89(1967),94-103.

[5] ——; Varieties defined by quadratic equations, C.I.M.E., (1969)-III, 29-100.