数理解析研究所講究録
第 240 巻 1975 年 131-138

Open Holomorphic Maps of Compact Complex Manifolds

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1. Introduction. Let V and W be connected compact complex manifolds. We say that two holomorphic maps f and g of V into W are equivalent if and only if there are an automorphism a of V and an automorphism b of W with $g = bfa$. We want to introduce a nice space structure into the set of all equivalent classes. It is a kind of moduli problems. However, easy examples show that it may not even be a Hausdorff space with respect to a natural topology. Thus it needs some restrictions on manifolds and holomorphic maps. In this lecture, our main restriction is that the holomorphic maps in the consideration are open holomorphic maps.

We formulate our results as follows.

According to Douady [1], the set $H(V, W)$ of all holomorphic maps of V into W is an analytic space whose underlying topology is the compact-open topology. Here, by an analytic space, we mean a reduced, Hausdorff, complex analytic space. We denote by $O(V, W)$ the set of all open holomorphic maps of V onto W. Then $O(V, W)$ is an open subvariety of $H(V, W)$ unless it is empty. Let $Aut(V)$ and $Aut(W)$ be the automorphism groups of V and W, respectively. They are complex Lie groups. Now, the product group $Aut(W) \times Aut(V)$ acts on $O(V, W)$ as follows:

$$(b, a, f) \in \mathrm{Aut}(W) \times \mathrm{Aut}(v) \times O(V, W)$$

$$\longrightarrow bfa^{-1} \in O(V, W) .$$

Applying Holmann's theorem [2], we get the following theorem:

Theorem 1. Assume that $\mathrm{Aut}(V)$ is compact. Then the orbit space $O(V, W) / (\mathrm{Aut}(W) \times \mathrm{Aut}(V))$ with the quotient topology is an analytic space.

As an example, we consider the following case:

$V = T$: a complex 1-torus ,

$W = \mathbb{P}^1$: the complex projective line .

Then $O(T, \mathbb{P}^1)$ is the set of all elliptic functions on T. We denote by $O_n(T, \mathbb{P}^1)$, $n = 2, 3, \cdots$, the set of all elliptic functions of order n. In this case, we have:

Theorem 2. The analytic space $O(T, \mathbb{P}^1) / (\mathrm{Aut}(\mathbb{P}^1) \times \mathrm{Aut}(T))$ is decomposed into the connected components as follows:

$$O(T, \mathbb{P}^1) / (\mathrm{Aut}(\mathbb{P}^1) \times \mathrm{Aut}(T)) = M_2 \cup M_3 \cup \cdots ,$$

where each $M_n = O_n(T, \mathbb{P}^1) / (\mathrm{Aut}(\mathbb{P}^1) \times \mathrm{Aut}(T))$, $n = 2, 3, \cdots$, is an irreducible normal analytic space of dimension $2n - 4$.

(M_2 is one point.)

In this lecture, we only give an outline of the proof of Theorem 1. Since we use the deep result of Holmann, everything is easy. On the other hand, the proof of Theorem 2 requires some facts on the local structure of the analytic space $H(V, W)$ and on the concrete construction of the analytic space $O(T, \mathbb{P}^1)$. (For the details, see [3]).

In Theorem 1, the assumption that $Aut(V)$ is compact can not be dropped. At the end of this lecture, we give a simple counter example.

2. Outline of the proof of Theorem 1. We see that Satz 19, [2], and the following proposition imply Theorem 1.

Proposition 1. Assume that $Aut(V)$ is compact. Then the map

$$\phi : (b, a, f) \in Aut(W) \times Aut(V) \times O(V, W) \longrightarrow (bfa^{-1}, f) \in O(V, W) \times O(V, W)$$

is a proper map.

In order to prove Proposition 1, we first prove:

Proposition 2. Let $\{f_\nu\}_{\nu=1,2,\cdots}$ and $\{g_\nu\}_{\nu=1,2,\cdots}$ be sequences of open holomorphic maps of V onto W converging to open holomorphic maps f and g, respectively. Assume that, for each ν, $\nu = 1, 2, \cdots$, there is an automorphism b_ν of W with $g_\nu = b_\nu f_\nu$. Then there is an automorphism b of W such that (1) $g = bf$ and such that (2) $\{b_\nu\}_{\nu=1,2,\cdots}$ converges to b.

Proof of Proposition 2. Let Q be an arbitrary point of W. Let $P \in f^{-1}(Q)$. We put $R = g(P)$. We show that R does not depend on the choice of $P \in f^{-1}(Q)$. Using the assumption that $f_1, f_2, \cdots$ and f are all open maps, we can easily show that there is a sequence $\{P_\nu\}_{\nu=1,2,\cdots}$ of points of V converging to P such that $f_\nu(P_\nu) = Q$, $\nu = 1, 2, \cdots$. We put $R_\nu = b_\nu(Q)$, $\nu = 1, 2, \cdots$. Then

$$R_\nu = b_\nu(Q) = b_\nu f_\nu(P_\nu) = g_\nu(P_\nu) .$$

Since $\{g_\nu\}_{\nu=1,2,\cdots}$ and $\{P_\nu\}_{\nu=1,2,\cdots}$ converges to g and P, respectively, $\{R_\nu\}_{\nu=1,2,\cdots}$ converges to $g(P) = R$. Since each $R_\nu = b_\nu(Q)$, $\nu = 1,2,\cdots$, does not depend on the choice of $P \in f^{-1}(Q)$, R does not either. We write $R = b(Q)$. Then b is a map of W into itself. Since $f_\nu = b_\nu^{-1} g_\nu$, $\nu = 1, 2, \cdots$, the condition is symmetric. Hence b is a bijection of W onto itself.

We show that b is a homeomorphism. We put

$$A = \{(P, Q, R) \in V \times W \times W \mid f(P) = Q,\ g(P) = R\} .$$

Then A is a closed complex submanifold of $V \times W \times W$ which is isomorphic to V. Let

$$\lambda : V \times W \times W \longrightarrow W \times W$$

be the projection map. Then, by the proper mapping theorem, $\lambda(A)$ is a closed irreducible subvariety of $W \times W$. Let λ_1 and λ_2 be the restrictions to $\lambda(A)$ of the first and the second projection maps: $W \times W \longrightarrow W$, respectively. Then λ_1 and λ_2 are bijective and $b = \lambda_2\lambda_1^{-1}$. Since $\lambda(A)$ is irreducible, λ_1 and λ_2 are homeomorphisms. Hence b is a homeomorphism.

We show that b is an automorphism of W. We put

$$D_f = \{P \in V \mid (df)_P \text{ has rank} < \dim W\} ,$$

$$E_f = f(D_f) ,$$

$$D'_f = f^{-1}(E_f) .$$

Then D_f and D'_f are proper closed subvarieties of V and E_f is a proper closed subvariety of W. Moreover,

$$f : V - D'_f \longrightarrow W - E_f$$

is a surjective holomorphic map of maximal rank at every point. Hence f has local holomorphic cross sections τ on $W - E_f$. Then, locally, $b = g\tau$. This shows that b is holomorphic on $W - E_f$. Since b is a homeomorphism of W, b is a holomorphic map on W. A similar argument shows that b^{-1} is also a holomorphic map. Hence $b \in \mathrm{Aut}(W)$.

By the construction, $g = bf$. Moreover, we can easily show that the sequence $\{b_\nu\}_{\nu=1,2,\cdots}$ converges to b. This completes the proof of Proposition 2.

Q. E. D. of Proposition 2.

Now, the proof of Proposition 1 is easy.

Proof of Proposition 1. Let K be a compact subset of $O(V, W) \times O(V, W)$. We show that $\phi^{-1}(K)$ is a compact subset of $\mathrm{Aut}(W) \times \mathrm{Aut}(V) \times O(V, W)$. It suffices to show that, for any sequence $\{(b_\nu, a_\nu, f_\nu)\}_{\nu=1,2,\cdots}$ of points of $\phi^{-1}(K)$, we can choose a subsequence converging to a point of $\phi^{-1}(K)$. By the assumption that $\mathrm{Aut}(V)$ is compact, we may assume that $\{a_\nu\}_{\nu=1,2,\cdots}$ converges to $a \in \mathrm{Aut}(V)$. We put $g_\nu = b_\nu f_\nu a_\nu^{-1}$, $\nu = 1, 2, \cdots$. Since K is compact, we may assume that $\{(g_\nu, f_\nu)\}_{\nu=1,2,\cdots}$ converges to $(g, f) \in K$. We put $h_\nu = f_\nu a_\nu^{-1}$, $\nu = 1, 2, \cdots$. Then $g_\nu = b_\nu h_\nu$, $\nu = 1, 2, \cdots$. The sequence $\{h_\nu\}_{\nu=1,2,\cdots}$ converges to $h = fa^{-1}$. By Proposition 2, there is $b \in \mathrm{Aut}(W)$

such that $g = bh = bfa^{-1}$ and such that $\{b_\nu\}_{\nu=1,2,\cdots}$ converges to b. Then $(b, a, f) \in \phi^{-1}(K)$ and the sequence $\{(b_\nu, a_\nu, f_\nu)\}_{\nu=1,2,\cdots}$ converges to (b, a, f).

Q. E. D. of Proposition 1.

3. A counter example. Let $\mathbb{P}^1$ be the complex projective line. Then $O(\mathbb{P}^1, \mathbb{P}^1)$ is the set of all non-constant rational functions. Note that $\mathrm{Aut}(\mathbb{P}^1)$ is not compact. We show that $O(\mathbb{P}^1, \mathbb{P}^1) / (\mathrm{Aut}(\mathbb{P}^1) \times \mathrm{Aut}(\mathbb{P}^1))$ with the quotient topology is not a Hausdorff space.

Let ξ be an inhomogeneous coordinate in $\mathbb{P}^1$. We consider a one-parameter family $\{f_\lambda\}_{\lambda \in \mathbb{C}}$ of rational functions defined as follows:

$$f_\lambda(\xi) = \xi^3 + \lambda\xi \ , \quad \lambda \in \mathbb{C} \ .$$

Then f_0 and f_1 are not equivalent (under the action of $\mathrm{Aut}(\mathbb{P}^1) \times \mathrm{Aut}(\mathbb{P}^1)$). In fact, arranging the ramification exponents, the types of the branch points of f_0 and f_1 are (3, 3) and (3, 2, 2), respectively. On the other hand, if $\lambda \neq 0$, then f_λ and f_1 are equivalent, for

$$b_\lambda f_\lambda a_\lambda^{-1} = f_1 \ ,$$

where b_λ, $a_\lambda \in \mathrm{Aut}(\mathbb{P}^1)$ are defined by

$$b_\lambda(\xi) = (1 / \sqrt{\lambda})^3 \xi ,$$

$$a_\lambda(\xi) = (1 / \sqrt{\lambda}) \xi .$$

Hence f_0 and f_1 can not be separated in $O(\mathbb{P}^1, \mathbb{P}^1) / (\mathrm{Aut}(\mathbb{P}^1) \times \mathrm{Aut}(\mathbb{P}^1))$.

References.

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[2] H. Holmann, Komplexe Räume mit komplexen Transformations-gruppen, Math. Ann. 150, 327-360(1963).

[3] M. Namba, Moduli of open holomorphic maps of compact complex manifolds, to appear.

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