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Differential forms and Stratifications (II).

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§0. 問題の説明. 代数及び解析多様体(variety)の理論に於いてstratificationのideaの基本的な重要性は周知である。与えられたvarietyをstratifyして考察する時,次の三段階が基本的に現われると思われる。

(I) 与えられたvarietyを各々のstratum及びstrataの間の諸関係が単純(典型的)である様にstratifyする。

(II) 与えられた問題に対し,個々のstratumに対して(求められている)結果をうる。

(III) (II)で得られた結果を'piece together'して与えられたvariety又はsubvarietiesに対して,所期の結果を得る。

(I),(II)の段階を与えられた問題の'局所化の段階'及び(III)の段階を(局所的結果に対しての)

'大域的段階'と呼びうると思われる。どの様なvarietyを単純(もしくは典型的)と見做すかは，与えられた問題の性質及びvarietyの把握の仕方によると思われる。

この論説及び同title(I)(9月：超曲面の孤立特異点seminar)で述べるのは，diff. formsの理論と(上記で説明した様な)stratificationsのideaを結び付け様とする事である：

Contents :

Theorem 6.1$_d$) (§7)…… 付録．　18頁

1°の内容は2°〜4°のResumé⊕ algebraic de Rham C.C.I. (divisor case)で，1975年1月号のJapan Acad. Proc.へ掲さいされるannouncementの原稿である。1°は9月のセミナーの講究録にも掲さいしたのであるが，2°〜4°で述べた事実及びこの論説のbasic ideasの説明になると思われるので再び掲さいする。(御寛容を乞う次第である。)

2°〜5°に於いての§＊は現在作成中の原稿(De Rham cohomologies and Stratifications)の都合上のものであって，少々不体裁ではあるが，これも御寛容を乞う次第である。

筆者の予定では，§1〜2は，殆んどこの論説中の形を最終的に用いる予定であり，§3及び§7は，draftsである。(数学的内容は変更しない積りである。)

尚§1〜§3は，一貫した内容であるが，§7は，この論説では，孤立した形になっている：9月のセミナーで，'normalized series of prestratified spaces'

の概念を述べ、その様なseriesの存在定理(与えられたgerms of varietiesについて)を述べた。§7は、9月のセミナーで述べた、'germs of varietiesに付随されたnormalized series'のconstructionのdetailsである。

尚9月のセミナーの講究録には、§4: Normal-ized series of prestratified spaces, 及び§6: Normalized series attached to germs of varietiesを掲さいしたので、途中に述べた、De Rham cohomologies and stratificationsの原稿は、local real analytic varietiesに関するelementary propertiesを含む§5を除いては、Diff. forms and Stratifications I, IIに見出される：筆者は'De Rham coh. and stratifications'(Complex analytic de Rham coh. theoriesの第I部)を近い内にどこかに発表したい希望であるが、この論説及び9月セミナーの論説は、上記titleの論文の'text book'の一種と見做されると思われる。

大雑把に言って，9月のセミナーの内容は，種々のcontextに於いて，段階(I)に相当し亦，この論説は段階(III)に相当する。亦段階(II)に相当する部分は，~~筆者~~のmanuscript（[7] in 1°）に見出されうるであろう。

筆者の現在の目的は，勿論 complex analytic varieties の de Rham coh. theory の'建設'である。この論説は，その為の'基礎工事'の積りであるが，一方この論説及び9月セミナーでの論説は上記の枠にはまらないと思われる：

① 'Normalized of prestr.' の概念は，'ramified covering の方法'(*) の方法の徹底化と stratifi-cations の方法の結合とみなされる：与えられた variety V に対して，一つの ramified map $\varphi: V \to V_1$ を考える代りに，Normalized series の様に系列 of ramified map

$$V \xrightarrow{\varphi} V_1 \xrightarrow{\varphi_1} V_2 \xrightarrow{\varphi_2} \cdots$$

を適当にとる事は（筆者の意見では）興味が

(*) 与えられた variety V に対して，ramified map $\varphi: V \to V'$ を適当に作り，V の性質の考察を（より簡単な）V' 及び φ の ramification locus の考察に転化する，という意味での

ある様に思われる。亦(上記の様な方法で考察されうる)対象が, de Rham coh. のみに限定される訳でないと言うのは,恐らく妥当性のある意見であろう。特に筆者が導入した'higher discriminant condition' は de Rham coh. の幾つかの quantitative properties の考察に於いて極めて有用であるが、この条件は同時に定性的でもある。この条件が, ramified map の系列の考察に(より広い範囲で)有用かどうかを調べるのは,筆者には特に興味がある。

2° この論説の主眼である, C.C.I. は,代数性の強い de Rham coh. theories に対しても,適用可能である事が期待される。(1° の最後の部分の例を参照) 亦 Residue theories とも極めて強い類似性がある様に思われる。(後述参照)

C.C.I. (Cochain complex with incidence conditions) について。

順序が多少逆になったが,この論説の主眼である C.C.I. の説明を簡単にする。この論説

中及び1°で述べた様に C.C.I. の概念の導入は、種々の de Rham cohomology 群の 'local' 及び 'global' な対象の関係を明らかにする希望を持ってなされた：$\mathbb{V}$ を top. space, $\mathcal{S}$ を $\mathbb{V}$ の prestratification とする。(2° 参照) この時 $\mathcal{S}$ から簡単な操作により三種類の sets：$\mathcal{S}_C$, $\mathcal{S}_O$, $\mathcal{S}_{CO}$ を定義する。

この時 $\mathcal{S}$ に attach された C.C.I. $\mathcal{C}$ とは、次の三種の collections よりなる。

$\{\mathcal{C}(X),\ X \in \mathcal{S}_C\}$, $\{\mathcal{C}(X),\ X \in \mathcal{S}_O\}$, $\{C(X),\ X \in \mathcal{S}_{CO}\}$. ここで $C(X)$ は cochain complex.

この時、幾つかの理由を持って、第一、第二の対象を夫々 'global' 及び 'local' な対象とみなす。そして第三の対象を 'global' 及び 'local' な対象を結び付ける (intermediate) object とみなす。(1°, 2°, 参照) そして C.C.I. $\mathcal{C}$ には、幾つかの exact sequences をみたす事が要請される：

1° (§1) は C.C.I. の axiomatic な一般論及び後に必要な概念の導入であり、一つの Lemma を含む：Lemma 1.1 は、'local objects' $H^*(\mathcal{C}(X))$, $X \in \mathcal{S}_O$ の性質から、'global objects' $H^*(\mathcal{C}(X))$ $X \in \mathcal{S}_C$

の幾つかの性質が導かれるものを示すものである。

§2 (3°) は、C.C.I. の例として、C^∞ de Rham. C.C.I. 及び singular C.C.I. の二つの概念を導入する。この二つは、容易に'理解'される性質のものであり、後の理論に応用される。亦 C.C.I. 一般の理解を助けると期待する。

§3 (4°) は 'structure' を持った C.C.I. の例であり、我々の主目的の一つ ([5], [7] in 1°) の key point の一つをなす。 細部は §3 に譲るとして、§3 の結果は、[5] で述べた 'C^∞-analogue of meromorphic comp-arison theorem' の考察に於いても我々の C.C.I. の概念が、有用であり亦更に (I)〜(III) で述べた段階に沿っての議論が可能である事を示す。事実 C.C.I. 一般の概念は、§3 の内容が motivation となった (1° 及び [7] in 1° 参照)。

更に 1° に述べた代数的な例も興味あり、これから考察の対象とするべきものと思われる。

この例から出る応用は次の通り：記号は 1° の通りとする：X : affine variety, D : hypersurface of V. この時.

$\omega \in H^*(\hat{\Omega}(V-D))$ が第2種 (w.r.t. (V, D)) とは,

$$\exists\, \omega' \in H^*(\hat{\Omega}(V)) \text{ such that } i^*(\omega') = \omega.$$

ここで i^* は $i: V-D \hookrightarrow V$ より引起される自然な homomorphism. この時

定理 ω: 第2種 w.r.t. $(V, D) \Leftrightarrow \rho^*(\omega) \in \{H^*(\hat{\Omega}(D, V-D))\} = 0$.

ここで $\rho^*: H^*(\hat{\Omega}(V-D)) \longrightarrow H^*(\hat{\Omega}(D, V-D))$ は,自然な homomorphism. (cf. 2°).

この内容は, ω が第2種か否かが, Dの充分小さな近傍での挙動で決定出来る事を示す. (これらの議論は, 微分形式の1, 2, 3種への分類への provisional arguments である).

更に C.C.I. の理解を助ける為に, Residue の理

論との関連を簡単に付す。以下では C^∞ theory について述べるが、いろいろの structure (analytic, algebraic ...) に関して述べるのも容易である。

M を C^∞-manifold, V を M の submanifold とし N を更に V の充分小さな適当な近傍とする。

この時、次の4種の coh. gr が考えられる:

$$H^*(M),\quad H^*(M-V),\quad H^*(V)\simeq H^*(N(V)),\quad H^*(N(V)\cap M-V)$$

最後の coh. group を $H^*(V, M-V)$ で表わす。

この時これらの4種の coh. gr. は、次の exact sequence (Mayer-Vietoris sequence) で関係づけられる:

$$① \quad \cdots \longrightarrow H^*(M) \longrightarrow H^*(M-V)\oplus H^*(V) \longrightarrow H^*(V, M-V) \longrightarrow \cdots$$

$H^*(M)$ を 'global object' 他の三種を 'local object' とみなすのは Residue theories に於いてなされる。①は Residue th. の一つの idea である。Residue th は①及び $H^*(M-V)$ (or $H^*(V, M-V)$) と $H^*(V)$ との関連を更に含むが、M, V が variety の場合、ここではその事については触れない。

上記の結果を我々の論法で記述すれば次の通りである。

$\mathcal{S} = M$ の prestratification $= \{V, M-V\}$,

$\mathcal{S}_C$ の最も本質的な元 $= M$,

$\mathcal{S}_{CO} = \phi\}$

$\mathcal{S}_O = \{V, M-V, (V, M-V)\}$

$\mathcal{C} = \{C^*(V), C^*(M-V), C^*(V, M-V), C^*(M)\}$,

ここで C^* は, singular cochain complex. そして $\mathcal{C}$ に課される条件は丁度 ① である.

以上の説明は, 少し簡単かも知れないが "$H^*(M)$ をわかる" 為には, "$H^*(M-V)$, $H^*(V)$, $H^*(V, M-V)$ の性質" が (ある種の　問題には) 充分である事を示すと言えるであろう.

以上の様に解釈した 'local - global' parts of Residue theory が我々の C.C.I. 導入の motivation の他の一つである。

終了.

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De Rham cohomologies and stratifications .

Complex analytic de Rham cohomology III.

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The importance of the idea of stratifying varieties in the study of algebraic and analytic varieties is well known. The investigation of stratification of varieties would involve basically the following steps*:

(1) To stratify varities so that each stratum as well as the relations among the strata ,e.g., incidence relation ,..., are of simple (or typical) forms.

(2) To obtain results of desired nature for each stratum or each series of strata,etc. with respect to a fixed stratification for given varieties.

(3) To piece together results from the step (2) in order to obtain results of a desired sort for given varieti--es and subvarieties,....

The steps {(1),(2)} and (3) might reasonably called , respectively, localization steps (for given global problems)

(*) See R.Thom [8] , H.Whitney [9]. The author learned the theories of stratifications in connection with his proposed approach to Complex analytic de Rham cohomology. (Cf. [4],[5].)

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and globalization steps (to be applied to local results).

In [5],[7] we investigated certain quantitative proper--ties of real analytic varieties. Results of [5] are used in our study of the complex analytic de Rham cohomology. Our investigations in [7] are carried out using steps (1), (2) and (3). Exact sequences of Mayer- Vietoris type are used repeatedly in our globalization steps.The basis of our arguments used in the globalization steps is algebraic in nature.

The main purpose of the present note is to introduce the notion of <u>cochain complex with incidence relations (C.C.I.) for a prestratified space</u>. (See n.1. and n.2. below.) The arguments used in the study of C.C.I. are gene--ralizations, as well as abstractions , of those in [7]. When C.C.I.'s are related to de Rham cohomologies of certain types, the arguements applicable to C.C.I.'s in general clarify relations between 'local' and 'global'(*) data in the de Rham cohomologies in question. Actually the author's hope in introducing the notion of C.C.I. is to clarify relations between 'local' and 'global' data in de Rham cohomologies of various types.

(*) The terms 'local' and 'global' in this note should be understood in the sense explained at the beginning of this note.

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The cotents of this note are preliminary in nature. However, the arguments applicable to C.C.I. in general are indispensable in [5] and have certain theoretically pleasa--nt aspects.

n.1. Prestratifications. Let X be a topological space. By a prestratification* Ⓢ of X we mean a collection $Ⓢ = \{S_\lambda\}_{\lambda \in \Lambda}$ of subsets S_λ's of X satisfying the following conditions.

$(1.1)_1$ X is the disjoint union of S_λ's in Ⓢ : $X = \bigcup S_\lambda$, $S_\lambda \in Ⓢ$.

$(1.1)_2$ For each stratum $S_\lambda \in Ⓢ$, the dimension : $\dim S_\lambda \in Z^+$ is given.

$(1.1)_3$ Frontier condition : For each stratum $S_\lambda \in Ⓢ$, $\mathrm{fron}(S_\lambda) = \overline{S}_\lambda - S_\lambda$ is the disjoint union of lower dimensi--onal strata of Ⓢ .

To substantially simplify notations in later arguments we assume that

$(1.1)_4$ Ⓢ is a finite set.**

A pair (X,Ⓢ) consisting of a topological space and its prestratification Ⓢ will be called a prestratified space. Let (X,Ⓢ) be a prestratified space. For Ⓣ ⊂ Ⓢ, let |Ⓣ| denote the

(*) For definitions of stratifications and prestratifi--cations , see J.Mather [2], R.Thom[8] . Our definition of prestratifications is ,for technical reasons, not the same as in [2],[8].

(**) For the case where Ⓢ is locally finite , see [6]

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support of $\mathbb{T}$: $|\mathbb{T}| = \bigcup S_\lambda$, $S_\lambda \in \mathbb{T}$. Moreover, for $\mathbb{T}' < \mathbb{T} < \mathbb{S}$, $\overline{|\mathbb{T}'|}_{\mathbb{T}}$ denotes the closure of $|\mathbb{T}'|$ in $|\mathbb{T}|$. We list certain notations used below. For $\mathbb{T} \subset \mathbb{S}$, define $\mathbb{T}_C, \mathbb{T}_m$ by

$$\mathbb{T}_C = \{\mathbb{T}' \subset \mathbb{T} : \overline{|\mathbb{T}'|}_{\mathbb{T}} - \mathbb{T}' = \phi \text{ or closed}\} ,$$

$$\mathbb{T}_m = \{S_\lambda \in \mathbb{T} : l(S_\lambda) \leq m\}.^*$$

Moreover, let $\mathbb{T}_0$ denote the collection of series of strata in $\mathbb{T}$:

$$\mathbb{T}_0 = \{ S_{\lambda_1,\ldots,\lambda_t} = S_{\lambda_1} < \cdots < S_{\lambda_t} \ : S_{\lambda_j} \in \mathbb{T} (j = 1,\ldots,t) \} .$$

In the above $S_{\lambda_1} < S_{\lambda_2}, \ldots$ means that $S_{\lambda_1} \subset \text{fron}(S_{\lambda_2})$, ... For $\mathbb{T}' \subset \mathbb{T}$ and a series $S_{\lambda_1,\ldots,\lambda_t}$, $S_{\lambda_j} \in \mathbb{T}'$, let $\mathbb{T}'_m(S_{\lambda_1,\ldots,\lambda_t})$ denote the intersection $\mathbb{T}'_m \cap \{S \in \mathbb{T}' : S_{\lambda_t} < S\}$. For $\mathbb{T} < \mathbb{S}$ define $\mathbb{T}_{0C}$ by

$$\mathbb{T}_{0C} = \{\mathbb{T}'_m(S_{\lambda_1,\ldots,\lambda_t}) : \mathbb{T}' < \mathbb{T} , S_{\lambda_j} \in \mathbb{T}' , j = 1,\ldots,t\} .$$

Then one easily derives the following fact .

(1.2) If $\mathbb{T} \in \mathbb{S}_C$, then $\mathbb{T}_m \in \mathbb{S}_C$ and $\mathbb{T}_C, \mathbb{T}_{0C} \subset \mathbb{S}_C$. Moreover, if $\mathbb{T}_1 \mathbb{T}_2 \in \mathbb{S}_C$ satisfy the relation $\mathbb{T}_1 \vee \mathbb{T}_2$, then $\mathbb{T}_1 \cup \mathbb{T}_2 \in \mathbb{S}_C$.

Here $T_1 \vee T_2$ if , for any $S_{\lambda_i} \in \mathbb{T}_i$ $(i = 1,2)$, $S_{\lambda_1} \not< S_{\lambda_2}, S_{\lambda_1} \not> S_{\lambda_2}$.

n.2. Cochain complex with incidence relation (C.C.I).

Let $\mathbb{R}$ be a noetherian ring , and let $(X, \mathbb{S})$ be a prestratified space. Moreover, let $\mathbb{C}(\mathbb{S})$ be a collection

(*) For $S \in \mathbb{S}$, the length $l(S)$ can be defined in an obvious manner (see [6]).

of assignments $\{\mathbb{C}'(\mathbb{S}),\mathbb{E}(\mathbb{S}),\mathbb{E}'(\mathbb{S})\}$ of the following forms: $\mathbb{C}'(\mathbb{S})$: $\mathbb{T}\in\mathbb{S}_C\to\mathbb{C}'(\mathbb{T})$, $\mathbb{E}(\mathbb{S})$: $\mathbb{U}\in\mathbb{S}_0\to\mathbb{E}(\mathbb{U})$ and $\mathbb{E}'(\mathbb{S}):\mathbb{U}\in\mathbb{S}_{0C}\to\mathbb{E}'(\mathbb{U})$. Here $\mathbb{C}'(\mathbb{T})$, $\mathbb{E}(\mathbb{U})$ and $\mathbb{E}'(\mathbb{U})$ are $\mathbb{R}$-cochain complexes. The collection $\hat{\mathbb{C}}(\mathbb{S})=\{\mathbb{C}'(\mathbb{S}),\mathbb{E}(\mathbb{S}),\mathbb{E}'(\mathbb{S})\}$ as above is called a *cochain complex with incidence relations attached to* $(X,\mathbb{S})$ (C.C.I. attached to $(X,\mathbb{S})$) if $\hat{\mathbb{C}}(\mathbb{S})$ is equipped with isomorphisms $i(S), i(\mathbb{T}_1,\mathbb{T}_2), i(\mathbb{U}_1)$ and homomorphisms(*) $h_k(\mathbb{T}_m), h_k(\mathbb{U}_m)$ of the following forms.

$(1.3)_1$ $\quad i(S)$: $0\to\mathbb{C}'(S)\to\mathbb{E}(S)\to 0$, $S\in\mathbb{S}$.

$(1.3)_2$ $\quad i(\mathbb{T}_1,\mathbb{T}_2)$: $0\to\mathbb{C}'(\mathbb{T}_1\cup\mathbb{T}_2)\to\mathbb{C}'(\mathbb{T}_1)\oplus\mathbb{C}'(\mathbb{T}_2)\to 0$, $\mathbb{T}_1,\mathbb{T}_2\in\mathbb{S}_C$ and $\mathbb{T}_1\vee\mathbb{T}_2$.

$(1.3)_2'$ $\quad i(\mathbb{U}_1)$: $0\to\mathbb{E}'(\mathbb{U}_1)\to\bigoplus_{\lambda_{t+1}}\mathbb{E}(S_{\lambda_1,\dots,\lambda_{t+1}})\to 0$, where $\mathbb{U}_1=\mathbb{T}_1(S_{\lambda_1},\dots,\lambda_t)$ with $\mathbb{T}\in\mathbb{S}_C$ and $S_{\lambda_j}\in\mathbb{T}$. Moreover, $S_{\lambda_{t+1}}\in\mathbb{U}_1$.

$(1.3)_3$ $\quad 0\to\mathbb{C}'(\mathbb{T}_{m+1})\xrightarrow{h_1(\mathbb{T}_m)}\mathbb{C}'(\mathbb{T}_m\oplus\bigoplus_{\lambda_{m+1}}\mathbb{C}'(S_{\lambda_{m+1}}))\xrightarrow{h_2(\mathbb{T}_m)}\bigoplus_{\lambda_{m+1}}\mathbb{E}'(\mathbb{T}_m(S_{\lambda_{m+1}}))\to 0$, where $\mathbb{T}\in\mathbb{S}_C$ and $\mathbb{S}_{\lambda_{m+1}}\in\mathbb{T}_{m+1}-\mathbb{T}_m$.

$(1.3)_3'$ $\quad 0\to\mathbb{C}'(\mathbb{U}_{m+1})\xrightarrow{h_1(\mathbb{U}_m)}\mathbb{E}(\mathbb{U}_m)\oplus\bigoplus_{\lambda_{t+1}^{m+1}}\mathbb{E}(S_{\lambda_1,\dots,\lambda_{t+1}^{m+1}})\xrightarrow{h_2(\mathbb{U}_m)}\bigoplus_{\lambda_{t+1}^{m+1}}\mathbb{E}(\mathbb{U}_m(S_{\lambda_{t+1}^{m+1}}))\to 0$, where $\mathbb{U}_m=\mathbb{T}_m(S_{\lambda_1},\dots,\lambda_t)$ and $\mathbb{U}_{m+1}=\mathbb{T}_{m+1}(S_{\lambda_1},\dots,\lambda_t)$. Moreover, $S_{\lambda_{t+1}^{m+1}}\in\mathbb{U}_{m+1}-\mathbb{U}_m$.

Postulated conditions of the existence of isomorphisms in $(1.3)_1$, $\{(1.3)_2,(1.3)_2'\}$ and homomorphisms in $(1.3)_3,(1.3)_3'$ will be called '*Identification condition*', '*Disjoint condition*'

(*) Isomorphisms and homomorphisms are those of R-cochain complexes.

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and 'Incidence condition' ('Mayer- Vietoris condition') respec--tively. The collection of isomorphisms $i(S)$,... and homomo--rphisms h_k's will be denoted by $K(C(S))$. When we emphasize the role of $K(C(S))$, we say that $C(S)$ is $K(C(S))$-C.C.I. . Let $C(S)$ and $K(C(S))$ be as above . Then isomorphisms $i(S)$,. ...and homomorphisms h_k's of cochain complexes induce corre--sponding isomorphisms $i^*(S)$,...and homomorphisms h_k^*'s of cohomology groups naturally. The collection of $i^*(S)$,...and h_k^*'s will be denoted by $K^*(C(S))$.

Equivalences between C.C.I.'s. Let (X,S) be a prestra--tified space, and let $C^i(S)$ be $K(C^i(S))$-C.C.I. $(i = 1,2)$. Moreover, let α^*, β^* and β'^* be families of R-homomorphisms of the following forms: $\alpha^* = \{\alpha^*(T) : H^*(C'^1(T)) \to H^*(C'^2(T)),$ $T \in S_C\}$, $\beta^* = \{\beta^*(U) : H^*(E^1(U)) \to H^*(E^2(U)), U \in S_0\}$ and β'^* $= \{\beta'^*(U) : H^*(E'^1(U)) \to H^*(E'^2(U)) , U \in S_{0C}\}$. Then we can define the notion of commutativity of $\{\alpha^*, \beta^*, \beta'^*\}$ with $K(C^i(S))$ in an obvious manner ([6]). We say that $C^i(S)$ $(i = 1,2)$ are $\{\alpha^*, \beta^*, \beta'^*\}$- equivalent if (i) $\{\alpha^*, \beta^*, \beta'^*\}$ commute with $K(C^i(S))$ $(i = 1,2)$ and if (ii) the homomorphism $\beta^*(U)$ is an isomorphism for any $U \in S_0$.

Now , in our investigations, there are reasons for regarding $C'(T)$ $,T \in S_C$ as 'global' data and $E(U)$ $,U \in S_0$ as 'local' data. The following lemma shows that certain propert-

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-ies of global data are derived from those of local data.

Lemma 1, Let $\mathcal{C}^i(\hat{S}) = \{C'^i(S), E^i(S), \mathcal{E}'^i(\hat{S})\}$ be $\mathcal{R}(\mathcal{O}^i(\mathcal{S}))$ -C.C.I. ($i = 0,1,2$) of a prestratified space $(X,\mathcal{S})$.

(I) If $H^*(\mathcal{E}^0(\mathcal{U}))$ is a finitely generated $\mathcal{R}$-module for each $\mathcal{U} \in \mathcal{S}_c$, them $H^*(C'^0(\mathcal{U}))$ is so for each $\mathcal{T} \in \mathcal{S}_c$.

(II) Let $\mathcal{C}^i(\mathcal{S})$ ($i = 1,2$) be $\{\alpha^*, \beta^*, \beta'^*\}$-equivalent . Then $\alpha^*(\mathcal{T})$ is an $\mathcal{R}$- isomorphism for each $\mathcal{T} \in \mathcal{S}_c$. Here α^*, β^*, β'^* are families of $\mathcal{R}$-hbmomorphisms of the forms given in the be- -ginning of n.2.

For the proof of Lemma 1, see [6].

n.3. An exact sequence of Mayer- Vietoris type. Let K be an algebraically closed field of any characteristic. In n.3. every variety in question is assumed to be a reduced K-variety . Let A^n, V and D be an affine space of dimension n, a variety in A and a divisor in A, respectively, such that for each irreducible component V_j of V , $V_j \not\subset D$ and $V_j \cap D \neq \phi$. We denote by W the variety in A characterized by $|W| = |V| \cap |D|$. Now let $\mathcal{O}, \mathcal{I}_V, \mathcal{I}_W$ and $\mathcal{I}_D = \mathcal{O}(h)$ denote respectively the ring $K[x_1,\ldots,x_n]$ and the ideals of V,W,and D. The completions $\varprojlim_n \mathcal{O}/\mathcal{I}_V^n$ and $\varprojlim_n \mathcal{O}/\mathcal{I}_W^n$ are denoted by $\hat{\mathcal{O}}^V$ and $\hat{\mathcal{O}}^W$ respectively. We denote the localizations $\mathcal{O}[h^{-1}]$ and $\hat{\mathcal{O}}^W[h^{-1}]$ by $\mathcal{O}[*D]$ and $\hat{\mathcal{O}}^W[*D]$ respectively. Moreover, let $\hat{\mathcal{O}}^{V-W}$ and $\hat{\mathcal{O}}^{W,V-W}$ be respe- -ctively the completions defined by $\varprojlim_n \mathcal{O}[*D]/\mathcal{O}[*D]\cdot\mathcal{I}_V^n$ and $\varprojlim_n \hat{\mathcal{O}}^W[*D]/\hat{\mathcal{O}}^W[*D]\cdot\mathcal{I}_V^n$. In the above we regard $\mathcal{I}_V$ as contained in $\mathcal{O}[*D]$ and $\hat{\mathcal{O}}^W[*D]$ in a natural fashion. For the graded ring Ω of K-differential forms over A, let $\hat{\Omega}^V, \hat{\Omega}^W, \hat{\Omega}^{W,V-W}$ respectively denote $\Omega\otimes\hat{\mathcal{O}}^V, \dots, \Omega\otimes\hat{\mathcal{O}}^{W,V-W}$. Then we have :

Lemma 2. For the rings $\hat{\Omega}^V, \dots, \hat{\Omega}^{W,V-W}$, the exact sequence

$$0 \to \hat{\Omega}^V \xrightarrow{\hat{\rho}^V_W \oplus \hat{\rho}^V_{V-W}} \hat{\Omega}^W \oplus \hat{\Omega}^{V-W} \xrightarrow{\hat{\rho}^W_{W,V-W} \oplus \hat{\rho}^{V-W}_{W,V-W}} \hat{\Omega}^{W,V-W} \to 0, \qquad \text{(I)}$$

holds. In (I) the continuous homomorphisms $\hat{\rho}^V_W, \dots, \hat{\rho}^{V-W}_{W,V-W}$ are determined naturally from topologies of $\hat{\mathcal{O}}^V, \dots \hat{\Omega}^{W,V-W}$.

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For the proof of Lemma 2, see [6]. The exact sequence (I) relates cohomology groups $H^*(\hat{\Omega}^V)$, $H^*(\hat{\Omega}^{V-W})$ and $H^*(\hat{\Omega}^{W,V-W})$ to the cohomology group $H^*(\hat{\Omega}^V)$. For a pair (V,W) of smooth varieties (defined over the complex number field $\mathbb{C}$), the idea of relating the cohomology groups of W,V-W and N(W)-W (*) to that of V may be regarded as one of the basic ideas in the classical (analytic) theories of residues. (Cf. J.Leray[2], P.A.Griffith[1],..) . The sequence (I) might be regarded as a generalization in an alge-braic direction of the idea explained above. Moreover, Lemma 2 enables us to attach the algebraic de Rham C.C.I. to the pre-stratified space $(V, \mathfrak{S}=(W,V-W))$ in a natural manner.

Remarks about results untouched here. In this note, we have spent several pages explaining ideas used in defining C.C.I.'s. For arguements on C.C.I.'s untouched here, see [6]. In particular , [6] contains examples of C.C.I.'s such

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(*) N(W) is a suitable neighbourhood of W in V.

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as the singular, the C^{∞}- de Rham, and the P.G.(polynomial growth) de Rham C.C.I.'s, as well as an application of the sequence (I).

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1. Cochain complexes with incidence relations.

1.1. Prestratified spaces. Let V be a topological sp--ace and S a subset of V. For a subset V' of V such that $S \subset V'$, we denote by $\overline{S}_{V'}$ the closure of S in V'. When it is clear from contexts that we are taking the closure of S in V' we write $S_{V'}$ simply as $\overline{S}$.

Now we define the notion of a prestratification of a topological spce in the following manner(*)

Definition 1.1. Let V be a topological space. A <u>prestra--tification</u> S of V is a collection as follows:

$(1.1)_1$ A family $S=\{S_\lambda; \lambda\in\Lambda\}$ of subsets S_λ's of V (call--ed <u>strata</u> of S).

$(1.1)_2$ A family $D=\{\dim_\lambda; \lambda\in\Lambda\}$ of integer valued functions $\dim_\lambda$'s (called <u>dimention functions</u> on S_λ's).

Families S and D are required to satisfy the following conditions.

$(1.2)_1$ S is locally finite.

$(1.2)_2$ V is expressed as the disjoint union of all the strata in S.

(*) For the generel notions of stratifications and pre--stratifications, see J.Mather [ ], R.Thom [ ] and H.Whitney [ ]. Our definition of prestratification is not the same as theirs by technical reasons. Our definition is convenient to treat prestratifications of topological spaces(like C^∞-mani--folds, real and complex analytic varieties, algebraic varie--ties,etc.) at the same time.

$(1.2)_3$ Dimension condition: For each $S_\lambda \in S$, the dimension function $\dim_\lambda(P_\lambda)$ is bounded on S_λ.

$(1.2)_4$ Frontier condition: For each $S_\lambda \in S$, the frontier: fron $S_\lambda = \overline{S}_\lambda - S_\lambda$ is expressed as the disjoint union: fron $S_\lambda = \cup S_\mu$; $S_\mu \in S$, $S_\mu \cap (\overline{S}_\lambda - S_\lambda) \neq \phi$. Moreover, for any $S_\mu \subset$ fron S_λ, $\max._{P_\mu} \dim_\mu(P_\mu) < \max._{P_\lambda} \dim_\lambda(P_\lambda)$. Here $P_\mu \in S_\mu$ and $P_\lambda \in S_\lambda$.

We call $\dim_\lambda(P_\lambda)$ the dimension of S_λ at P_λ with respect to S and write it usually as $\dim_{P_\lambda} S_\lambda$. Moreover, we call $\max_{P_\lambda} \dim_{P_\lambda} S_\lambda$ the dimension of S_λ with respect to S and write it as $\dim S_\lambda$. When there is no fear of confusions, we call the dimension of S_λ at P_λ (the dimension of S_λ) with respect to S simply the dimension of S_λ at P_λ (the dimension of S_λ).

In the sequel of this papar, we always assume that a prestratification S of a topological space satisfies the following condition.

(1.3) There exists a positive number n_0 such that

$$\dim S_\lambda \leqq n_0 \text{ for each stratum } S_\lambda \text{ of } S .$$

Remark : Let $S = (S,D)$ be a prestratification of a topological space V. When there is no fear of confusions, we write the prestratification S as S. Also we call a stratum S_λ of S as a stratum S_λ of S . Of course these simplified notations coincide with the standard notations (cf. J.Mather [ ].)

Let V be a topological space, and let $S = (S,D)$ be a prestratification of V. Let $S_{\lambda_1} \neq S_{\lambda_2}$ be strata of S. We write $S_{\lambda_1} \prec S_{\lambda_2}$ if $S_{\lambda_1} \subset$ fron S_{λ_2}. For a sequence $S_{\lambda_1} \prec \dots \prec S_{\lambda_t}$ of strata of S , we use an abbriviated notation: $S_{\lambda_1,\dots,\lambda_t}$. Moreover, let $S_\lambda \in S$ and $l(S_\lambda)$ the positive number characteri-zed as follows: There exists a sequence of the form: $S_{\lambda_1,\dots,\lambda_l}$. But there does not exsist a sequence of the form: $S_{\lambda_1,\dots,\lambda_{l+1}}$. Here $S_{\lambda_1} = S_\lambda$ and $l = l(S_\lambda)$. This integer $l(S)$ will be called <u>the length of S_λ in</u> S. Note that $l(S_\lambda) \leqq \dim S - \dim S_\lambda + 1$.*
Furthermore, let $T_i (i = 1,2)$ be subsets of S. We say that T_1 and T_2 are <u>independent</u> if, for any $S_{\lambda_i} \in T_i$ (i=1,2), $S_{\lambda_1} \not\subset \overline{S}_{\lambda_2}$, $S_{\lambda_2} \not\subset \overline{S}_{\lambda_1}$. We use the symbol $T_1 \vee T_2$ to mean that T_1, T_2 are independent.

Let V be a topological space and $S = (S,D)$ a prestra-tification of V. Moreover, let V_1 be an open set of V. Defi-ne a collection S_1 of subsets of V_1 by $S_1 = \{S_{\lambda_1} \cap V_1\}_{\lambda_1}$. Here strata S_{λ_1} exhaust all the strata of S such that $S_{\lambda_1} \cap V_1 \neq \phi$.

(*) $\dim S = \max_\lambda \dim S_\lambda$.

Moreover, define a collection D_1 of integer functions by $D_1 = \{\dim^1_{\lambda_1}\}_{\lambda_1}$. Here $\dim^1_{\lambda_1}$ is the restriction of $\dim_{\lambda_1}$ to $S_{\lambda_1} \cap V_1$. Then it is clear that $\mathbf{S}_1 = (S_1, D_1)$ is a prestratification of V_1 in an obvious manner. We call this prestratification $\mathbf{S}_1$ the <u>restriction</u> of $\mathbf{S}$ to V_1.

A pair $(V,\mathbf{S})$ of a topological space V and a prestratification $\mathbf{S}$ of V will be called a <u>prestratified space</u>. Let $(V_i,\mathbf{S}_i)(i=1,2)$ be prestratified spaces, and let $f: V_1 \to V_2$ be a continuous map. We say that f is <u>compatible</u> with $\mathbf{S}_i (i=1,2)$ if, for each stratum S_{λ_1} of $\mathbf{S}_1$, $f(S_{\lambda_1})$ is a stratum of $\mathbf{S}_2$. A continuous map $f: V_1 \to V_2$, which is compatible with $\mathbf{S}_i (i=1,2)$, will be called a <u>map from the prestratified space</u> $(V_1,\mathbf{S}_1)$ <u>to</u> $(V_2,\mathbf{S}_2)$.

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<u>C, O and CO-sets of prestratification</u>*. Let $(V,\mathbf{S})$ be a prestratified space. We write $\mathbf{S}$ explicitly as $\mathbf{S} = (S,D)$. For a subset $T \subset S$, let $|T|$ denote the support: $\bigcup S_{\lambda_j}$, $S_{\lambda_j} \in T$, of T. Note that, for a subset $T \subset S$, $|T|$ is closed if and only if

$(1.3)_1$ for any $S_\lambda \in T$, $\text{fron}\, S_\lambda \subset |T|$.

Moreover, for a subset $T \subset S$,

$(1.3)_2$ $\overline{|T|} - |T| = \bigcup S_\mu$, where S_μ's are strata of S such that $S_\mu \subset \text{fron}\, S_\lambda$ with a stratum $S_\lambda \in T$ and $S_\mu \notin T$.

Furthermore, let T be a subset of S. Assume that $\overline{|T|} - |T|$

(*) The letters C and O are taken from the initials of the words 'closed' and 'order'.

$\neq \phi$. Then $\overline{|T|} - |T|$ is closed if and only if

$(1.3)_3$ for any S_μ such that $S_\mu \in \overline{|T|} - |T|$, fron $S_\mu \cap T = \phi$.

Now we shall introduce certain notations convenient for later arguments: Let (V,S) be a prestratified space, where $S = (S,D)$. Let S_C denote the collection $\{T \subset S;\ \overline{|T|} - |T| = \phi$ or clo--sed$\}$. We call this set S_C the <u>C-set</u> of S. Moreover, we call the collection $\{S_{\lambda_1,\dots,\lambda_t}\}_{\lambda_1,\dots,\lambda_t}$ of sequences of strata in S the <u>O-set</u> of S and write it as S_O. Furthermore, we define the <u>CO-set</u> S_{CO} of S to be the subset of the product $S_C \times Z^+ \times S_O$ chara--cterized as follows: $(T, m, S_{\lambda_1,\dots,\lambda_t}) \in S_C \times Z^+ \times S_O$ is in S_{CO} if and only if $S_{\lambda_j} \in T (j=1,\dots,t)$. Now let $T \in S_C$, and let $m \in Z^+$. We denote by T_m the subset $\{S_\lambda \in T;\ l_T(S_\lambda) \leq m\}$ of T. Here $l_T(S_\lambda)$ is the length of S_λ in T. (Note that T defines a prestratifica--tion of $|T|$ of the form (T, D_T), where D_T is the restriction of D to T.) Moreover, let $T \in S_C$, and let $S_{\lambda_1,\dots,\lambda_t}$ be a sequ--ence of strata of T. Then we denote by $T(S_{\lambda_1,\dots,\lambda_t})$ the sub--set $\{S_\mu,\ S_{\lambda_t} \prec S_\mu\}$ of T. Note that $T(S_{\lambda_1}) = T(S_{\lambda_1,\lambda_2}) = \cdots = T(S_{\lambda,\dots,\lambda})$.

Now the following are derived easily from the above definitions.

$(1.4)_1$ Let $T_1, T_2 \in S_C$ such that $T_1 \vee T_2$. Then $T_1 \cup T_2 \in S_C$.

$(1.4)_2$ Let $T \in S_C$. Then subsets T_m, $T(S_\lambda) \in S_C$, where $m \in Z^+$ and $S_\lambda \in T$. Moreover, the C-set T_C of the prestratifi-

-cation (T,D_T) of T is contained in S_C.

Let $(T,m,S_{\lambda_1},\ldots,\lambda_t)\in S_{CO}$. Then T_m and $T_m(S_{\lambda_1},\ldots,\lambda_t)\in S_C$ by $(1.4)_2$. We define a map $h_{CO}: S_{CO}\longrightarrow S_C$ by $h_{CO}(T,m,S_{\lambda_1},\ldots\lambda_t) = T_m(S_{\lambda_1},\ldots,\lambda_t)$. When there is no fear of confusions, we wr-ite $(T,m,S_{\lambda_1},\ldots,\lambda_t)$ as $T_m(S_{\lambda_1},\ldots,\lambda_t)$.*

Remark: When a prestratification S of a topological space V is obtained from certain geometric considerations, the C-set S_C of S has clear geometric meanings. Moreover, if we want to investigate the C-set S_C in details, the O- and CO-sets S_O and S_{CO} appear in a natural fashion. (Cf.[ ],[ ]. Also see the end of § 2.) Furthermore, there are reasons to regard objects related to S_C and objects related to S_O, resp-ectively, as global and local objects(in the senses expla-ined in the introduction) in investigations of S. (Cf.[ ], [ ]) Objects related to S_{CO} can be, then, regarded as objec-ts connecting local and global data in the above senses.

1.2. Cochain complex with incidence relations (C.C.I.).

Let R be a noetherian ring. In this subsection §1.2., we fix R once and for all. Our arguments in §1.2. will be do-

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(*) Note that the map h_{CO} is (in general) not injective. However, the symbol $T_m(S_{\lambda_1},\ldots,\lambda_t)$ contains three symbols $T,m,S_{\lambda_1},\ldots,\lambda_t$ which characterize the element $(T,m,S_{\lambda_1},\ldots,\lambda_t)\in S_{CO}$. There-fore our use of the symbol $T_m(S_{\lambda_1},\ldots,\lambda_t)$ for $(T,m,S_{\lambda_1},\ldots,\lambda_t)$ will not be harmful.

-ne for this fixed ring R. We mean, in this subsection § 1.2, by a cochain complex C an R-cochain complex of the form: $C=\sum_n C_n$ where $C_n = 0$ for $n<0$ or $n>n_0$ with a suitable n_0.

Now we shall introduce the following

Definition 1.2. Let (V,S) be a prestratified space, whe-re S = (S,D). A <u>cochain complex C with incidence relations attached to S</u> is a pair (C,K) as follows:

$(1.5)_1$ C is a collection of cochain complexes of the following form: $C = \{C(X), X \in S_C, S_O \text{ or } S_{CO}\}$. Here C(X) is a cochain complex and $S_C, \ldots,$ are the C-set of $S, \ldots$.

$(1.5)_2$ K is a collection of homomorphisms of the follow-ing forms:

$(1.5)_{2.1}$ $K(S_\lambda)$: $C(\{S_\lambda\}) \longrightarrow C(S_\lambda)$; $S_\lambda \in S$. Here we regard $\{S_\lambda\}$ and S in the image $C(S_\lambda)$ respectively as elements of S_C and S_O.

$(1.5)_{2.2}$ $K(T_1,T_2)$: $C(T_1 \cup T_2) \longrightarrow C(T_1) \oplus C(T_2)$, where T_1, $T_2 \in S_C$ such that $T_1 \vee T_2$.

$(1.5)'_{2.2}$ $K(X_1)$: $C(X_1) \longrightarrow \oplus\, C(S^1_{\lambda_{t+1}})$, where X_1 is an element of S_{CO} of the form $(T,1,S_{\lambda_1},\ldots,\lambda_t)$ and $S^1_{\lambda_{t+1}}$'s exhaust all the strata in $T_1(S_{\lambda_1},\ldots,\lambda_t)$.

$$(1.5)_3 \quad \begin{cases} K_1(T,m):\ C(T_{m+1}) \longrightarrow C(T_m) \oplus \{\oplus C(S_{\lambda^{m+1}})\}, \\ K_2(T,m):\ C(T_m) \oplus \{\oplus C(S_{\lambda^{m+1}})\} \longrightarrow \oplus C(T_m(S_{\lambda^{m+1}})), \end{cases}$$

where $(T,m) \in S_C \times Z^+$ such that $T_{m+1} - T_m \neq \phi$. Moreover, $S_{\lambda^{m+1}}$ exha--ust all the elements in $T_{m+1} - T_m$.

$$(1.5)_3' \quad \begin{cases} K_1(X_m):\ C(X_{m+1}) \longrightarrow C(X_m) \oplus \{\oplus C(S_{\lambda_1}, \ldots, {}_{\lambda_{t+1}^{m+1}})\} \\ K_2(X_m):\ C(X_m) \oplus \{\oplus C(S_{\lambda_1}, \ldots, {}_{\lambda_{t+1}^{m+1}})\} \longrightarrow \oplus C(X_m(S_{\lambda_{t+1}^{m+1}})), \end{cases}$$

where $X_m = T_m(S_{\lambda_1}, \ldots, {}_{\lambda_t})$, $X_{m+1} = T_{m+1}(S_{\lambda_1}, \ldots, {}_{\lambda_t})$ such that $X_{m+1} - X_m \neq \phi$. Moreover, $S_{\lambda_{t+1}^{m+1}}$ exhaust all the atrata in $X_{m+1} - X_m$.

The data C and K are required to satisfy the following exact sequences:

$$(1.6)_1 \quad 0 \longrightarrow C(\{S_\lambda\}) \xrightarrow{K(S_\lambda)} C(S_\lambda) \longrightarrow 0.$$

$$(1.6)_2 \quad 0 \longrightarrow C(T_1 \cup T_2) \xrightarrow{K(T_1,T_2)} C(T_1) \oplus C(T_2) \longrightarrow 0.$$

$$(1.6)_2' \quad 0 \longrightarrow C(X_1) \xrightarrow{K(X_1)} \oplus C(S_{\lambda_{t+1}^1}) \longrightarrow 0.$$

$$(1.6)_3 \quad 0 \longrightarrow C(T_{m+1}) \xrightarrow{K_1(T,m)} C(T_m) \oplus \{\oplus C(S_{\lambda^{m+1}})\} \xrightarrow{K_2(T,m)} \oplus C(S_{\lambda^{m+1}}) \longrightarrow 0.$$

$$(1.6)_3' \quad 0 \longrightarrow C(X_{m+1}) \xrightarrow{K_1(X_m)} C(X_m) \oplus \{\oplus C(S_{\lambda_1}, \ldots, {}_{\lambda_{t+1}^{m+1}})\} \xrightarrow{K_2(X_m)} \oplus C(X_m(S_{\lambda_{t+1}^{m+1}})).$$

In the above exact sequences, the symbols $S, T_1, T_2, \ldots, X_1, \ldots, S_{\lambda^{m+1}}, S_{\lambda^{m+1}_{t+1}}, \ldots$ have the same meanings as in (1.5).

The conditions of the validity of the exact sequences $(1.6)_1, (1.6)_{2,2'}$ and $(1.6)_{3,3'}$ imposed on the pair (C,K) will be, respectively, called <u>the identity, the disjoint and the incidence (or the Mayer-Vietoris) conditions</u>.

Let (V,S) be a prestratified space, and let C =(C,K) be a cochain complex with incidence relations attached to S. We call C simply a <u>C.C.I. attched to</u> S.

<u>Remark on the terminology of C.C.I.</u> Let (V,S) be a pres-tratified space, and let C = (C,K) be a C.C.I. attached to S. We write S and C explicitly as S = (S,D) and $C = \{C(X); X \in S_C, S_0$ or $S_{CO}\}$. Note that our notion of C.C.I. is based on the <u>collection</u> C of cochain complexes and that C itself has <u>not</u> been endowed with a structure of a <u>cochain complex.</u> Now we form a direct sum $\oplus C(X)$, $X \in S_C, S_0$ or S_{CO}. The direct sum $\tilde{C} = \oplus C(X)$ is a <u>cochain complex</u> in a natiral manner; It is easy to form a collection $\tilde{K}$ of homomorphisms in terms of $\tilde{C}$ so that $\tilde{K}$ corresponds to K. Also it is easy to state conditions corresponding to (1.6) in terms od the pair $(\tilde{C}, \tilde{K})$. The above brief explanation would suffice to assure the existence of an equivalent form of the notion of C.C.I.based on the cochain complex $\tilde{C}$, and would also justify our use of the

C.C.I. (cochain complex with incidence relations) for the notion introduced in Definition 1.2.

Remark. Typical examples of C.C.I.'s will be given in §2, §3 and Appendix I: The singular and the C^{∞}-de Rham C.C.I.'s introduced in §2 would increase understandings of the notion of C.C.I.'s in general. On the otherhand, arguments concerned with the P.G.-C^{∞}-de Rham C.C.I. in §3 and those concerned with residue theories in Appendix I would make clear our motivations for the introduction of the notion of C.C.I.

Notions and notations related to C.C.I.. We will introduce certain notions and notations related to the notion of C.C.I:

(I) Let (V,S) be a prestratified space, where S = (S,D). Moreover, let C = (C,K) be a C.C.I. attached to S. We write C explicitly as $C = \{C(X),\ X \in S_C, S_{O_{\pm}} \text{ or } S_{CO_{\pm}}\}$.

$(I)_1$ The cochain complex C(X) of C for X: Let $X \in S_C, S_{O_{\pm}}$ or $S_{CO_{\pm}}$. We call C(X) the cochain complex of C for X.

$(I)_2$ The D-set S_D of S, the cochain complex C(Y) of C for Y and the homomorphism K(Y) of K for $Y \in S_D$:

To introduce terminologies similar to those in $(I)_1$, it is convenient to introduce the D-set S_D of S to be the collection as follows: (i) $S_\lambda \in S$. (ii) $(T_1\ T_2) \in S_C \times S_C$ such th--at $T_1 \vee T_2$. (iii) Elements X_1's in S_C, where X_1 is of the form: $X_1 = T_1(S_{\lambda_1}, \ldots, _{\lambda_t})$. (iv) $(T,m) \in S_C \times Z^+$. (iv)' $(T,m,m+1) \in S_C \times Z^+ \times Z^+$. (v) $X_m = T_m(S_{\lambda_1}, \ldots, _{\lambda_t}) \in S_{C\alpha}$, (v)' $(X_m, m+1) \in S_{C\alpha} \times Z^+$.

An element in S_D will be usually denoted by Y. Now we define, for $Y \in S_D$, <u>the cochain complex $C(Y)$ of C for Y</u> and <u>the homomorphism $K(Y)$ of K for Y</u> as follows:

$(\widetilde{\text{i}})$ If $Y = S_\lambda$, $C(Y) = C(\{S_\lambda\})$, $K(Y) = K(S_\lambda)$.

$(\widetilde{\text{ii}})$ If $Y = (T_1, T_2)$, $C(Y) = C(T_1 \cup T_2)$, $K(Y) = K(T_1, T_2)$.

$(\widetilde{\text{iii}})$ If $Y = X_1$, $C(Y) = C(X_1)$, $K(Y) = K(X_1)$.

$(\widetilde{\text{iv}})$ If $Y = (T,m)$, $C(T,m) = C(T_m)$, $K(T,m) = K_1(T,m)$.

$(\widetilde{\text{iv}})'$ If $Y = (T,m,m+1)$, $C(T,m,m+1) = C(T_m) \oplus \{\oplus\, C(S_{\lambda^{m+1}})\}$, $K(Y) = K_2(T,m)$

$(\widetilde{\text{v}})$ If $Y = X_m$, $C(Y) = C(X_{m+1})$ $K(Y) = K(X_m)$

$(\widetilde{\text{v}})'$ If $Y = (X_m, m+1)$, $C(Y) = C(X_m) \oplus \{\oplus\, C(S_{\lambda_1}, \ldots, _{\lambda_{t+1}^{m+1}})\}$, $K(Y) = K_2(X_m)$.

In the aboves, $S, (T_1, T_2), \ldots, (X_m, m+1)$ have the same meanings as in the definition of S_D. (Also see the exact sequences in (1.6)). Moreover, for $X_m = T_m(S_{\lambda_1}, \ldots, _{\lambda_t})$ in (v), (v)', we den--ote $T_{m+1}(S_{\lambda_1}, \ldots, _{\lambda_t})$ by X_{m+1}.

The above definitions are somewhat lengthy. However, arguments in later will be simplified by using the above definitions.

$(I)_3$ The restriction of C.C.I: Let $T \in S_C$. Then, in view of (1.4), the cochain complex $C(X')$ is defined for any $X' \in T_C$, T_O or T_{CO}. We write the collection $\{C(X'), X' \in T_C, T_O$ or $T_{CO}\}$ as C_T and call it the restriction of C to T. Moreover, let $Y' \in T_D$. Then, in view of (1.4), K(Y) is defined. We write the collection $\{K(Y'), Y' \in T_D\}$ as K_T and call it the restriction of K to T. Then the pair $C_T = (C_T, K_T)$ is a C.C.I. attached to the prestratificatoon (T, D_T) of T in an obvious manner. (D_T is the restriction of the collection D of dimension functions to T). We call C_T the restriction of C to T.

Remark. Let (V,S) be a prestratified space. Moreover, let $C = \{C(X), X \in S_C, S_O$ or $S_{CO}\}$ be a collection of cochain complexes C(X)'s. In this case we also call C(X) the cochain complex of C for X. Furthermore, let Y be an element in the D-set S_D. Then we can define a cochain complex C(Y) in an entirely same manner as in the case of a C.C.I. (Cf. (i)$\sim$(v)' in $(I)_2$.) We call this cochain complex C(Y) the cochain complex of C for Y.

(II) Sub C.C.I: Let (V,S) be a prestratified space, and let C = (C,K) be a C.C.I. attached to S. We write C explicitly as $C = \{C(X), X \in S_C, S_O$ or $S_{CO}\}$. Moreover, let $C' = (C', K')$ be an another C.C.I. attached to S. We say that C' is a sub-C.C.I. of C if

$(1.7)_1$ for any X S_C, S_0 or S_{CO}, $C'(X)$ is a subcochain complex of C(X),

and if

$(1.7)_2$ for any Y S_D, $K'(Y)$ is the restriction of K(Y) to $C'(Y)$.

Next let $C'' = C''(X)$, X S_C, S_0 or S_{CO} be a collectoon o cochain complexes $C''(X)$'s. We assume that $C''(X)$ is a subcoch-ain complex of C(X) for each X S_C, S_0 or S_{CO}. For each Y S_D, $C''(Y)$ denotes

the cochain complex of C'' for Y. Let K(Y) be the homomorphism of K for Y. We say that K(Y) preserves $C''(Y)$ if the following inclusions are valid according to the nature of Y.

$$K(S)(C''(S))\subset C''(S),\ K(T_1,T_2)(C''(T_1\ T_2))\subset C''(T_1)\oplus C''(T_2)$$
$$,\ldots,\ K(X_m,m+1)\left(C''(X_m)\oplus\{\oplus C''(S_{\lambda^{m+1}_{t+1}})\}\right)\subset\oplus C''(X_m(S_{\lambda^{m+1}_{t+1}})).$$

(Cf. (1.6)). In the aboves $S, T_1, T_2, \ldots$ have the same meanongs as in the definition of S_D.

We say that K preserves C'' if, for each $Y\in S_D$, K(Y) pre--serves $C''(Y)$.

(III) Equivalence of C.C.I.'s: Let (V,S) be a prestratif--ied space. Moreover, let $C_i = (C_i, K_i)$ (i=1,2) be C.C.I.'s attached to S. We write C_i, K_i (i=1,2) explicitly as $C_i = \{C_i(X),\ X\in S_C, S_0 \text{ or } S_{CO}\}$ and $\{K_i(Y),\ Y\in S_D\}$. For each $Y\in S_D$, we denote by $K_i^*(Y)$ the homomorphism (defined on $H^*(C_i(Y))$) induced from $K_i(Y)$.

$(\mathrm{III})_1$ Let $\alpha^* = \{\alpha^*(X),\ X \in S_C, S_0$ or $S_{CO}\}$ ($\alpha = \{\alpha(X),\ X \in S_C, S_0$ or $S_{CO}\}$) be a collection of homomorphisms $\alpha^*(X): H^*(C_1(X)) \longrightarrow H^*(C_2(X))$ ($\alpha(X): C_1(X) \longrightarrow C_2(X)$). For any $Y \in S_D$, we can form a homomorphism $\alpha^*(Y)$ ($\alpha(Y)$) defined on $H^*(C_1(Y))$ ($C_1(Y)$) by operating homomorphisms in α^* (α) on summands of $H^*(C_1(Y))$ ($C_1(Y)$). Moreover, let $\tilde{C}$ be a direct summands of cochain complexes in C. Then we can form also a homomorphism $\alpha^*_{\tilde{C}}$ ($\alpha_{\tilde{C}}$) defined on $H^*(\tilde{C})$ ($\tilde{C}$) by operating homomorphisms in α^* (α) on summands of $H^*(C)$ (C), If $\tilde{C}$ is of the form $C_1(Y)$ or $C_1(X)$, then $\alpha^*_C = \alpha^*(C_1(Y))$ or $\alpha^*_{\tilde{C}} = (C_1(X)), \ldots$

Now we say that α^* (α) <u>commutes with</u> K_1 <u>and</u> K_2 if for any Y S_D,

$$(1.8)^* \qquad \alpha^*_{\tilde{C}(Y)} \cdot K_1^*(Y) = K_2^*(Y)\, \alpha^*(Y),$$

$$(1.8) \qquad \alpha_{\tilde{C}(Y)} K_1(Y) = K_2(Y)\, \alpha(Y).$$

Here $\tilde{C}(Y)$ is the cochain complex appearing in the right side of the exact sequence in (1.6), in which the cochain complex $C_1(Y)$ appears: Namely, according to $Y = S$, $(T_1, T_2), \ldots, X_m$, $(X_m, m+1)$, $\tilde{C}(Y) = C_1(S)$, $C(T_1) \oplus C(T_2), \ldots, C_1(X_m) \oplus \{\oplus C_1(S_{\lambda_1}, \ldots, {}_{\lambda_t^m})\}$ $, \oplus C(X_m(S_{\lambda_{t+1}^{m+1}}))$, where $S, T_1, T_2, \ldots, X_m, (X_m, m+1)$ have the same meanings as in (1.6).

(III)$_2$ Let $\alpha^*=\{\alpha^*(X);\ X\in S_C, S_O$ or $S_{CO}\}$ be a collection of homomorphisms $\alpha^*(X): H^*(C_1(X)) \to H^*(C_2(X))$. We say that $\mathcal{C}_i=(C_i,K_i)(i=1,2)$ are α^*-equivalent if

(1.9)$_1$ α^* commutes with K_1, K_2,

and

(1.9)$_2$ for each $X\in S_O$, $\alpha^*(X)$ is bijective.

We say that $\mathcal{C}_i=(C_i,K_i)(i=1,2)$ are equivalent if they are α^*-equivalent with a suitable collection α^* of homomorphism as above.

(III)$_3$ Let $\alpha^*=\{\alpha^*(X)\}(\alpha=\{\alpha(X)\})$, $X\in S_C, S_O$ or S_{CO}, be a collection of homomorphisms $\alpha^*(X): H^*(C_1(X)) \to H^*(C_2(X))$ $(\alpha(X): C_1(X)\to C_2(X))$. Moreover, let $T\in S_C$. We denote by $\mathcal{C}_{iT}=(C_{iT},K_{iT})(i=1,2)$ the restrictions of $\mathcal{C}_i=(C_i,K_i)$ to T and by $\alpha^*_T(\alpha_T)$ the restrictions of $\alpha^*(\alpha)$ to the collection of all the elements in T_C, T_O or T_{CO}.

Now it is clear that if $\alpha^*(\alpha)$ commutes with K_1 and K_2 then $\alpha^*_T(\alpha_T)$ commutes with K_{1T} and K_{2T}. Also it is clear that if $\mathcal{C}_i=(C_i,K_i)(i=1,2)$ are α^*-equivalent, then $\mathcal{C}_{iT}=(C_{iT}, K_{iT})$ are α^*_T-equivalent.

A lemma connecting local and global data. Let (V,S) be a prestratified space. Moreover, let $\mathcal{C}_i=(C_i,K_i)(i=0,1,2)$ be

C.C.I.'s attached to S. Furthermore, let $\alpha^* = \{\alpha^*(X),\ X \in S_C, S_0$ or $S_{C0}\}$ be a collection of homomorphisms $\alpha^*(X): H^*(C_1(X)) \longrightarrow H^*(C_2(X))$. Now we prove a lemma which shows that certain pro--perties of

the cohomology gruups $H^*(C_i(X)), X \in S_C$ are derived from those of $H^*(C_i(X)),\ X \in S_0 (i=0,1,2)$.

Lemma 1.1. <u>Let $(V,S), C^i (i=0,1,2)$ and α^* be as above.</u> <u>We assume that S is a finite set.</u>

$(1.10)_1$ <u>If $H^*(C_0(X))$ is a finitely generated R-module for each $X \in S_0$, then $H^*(C_0(X))$ is so for each</u> $X \in S_C$.

$(1.10)_2$ <u>If $C_i = (C_i, T_i)(i=1,2)$ are α^*-equivalent, then</u>

$$\alpha^*(X): H^*(C_1(X)) \longrightarrow H^*(C_2(X))$$

<u>is bijective for each</u> $X \in S_C$.

We remarked that $H^*(C_i(X)),\ X \in S_C$ and $H^*(C_i(X)), X \in S_0$ $(i=0,1,2)$ can be regarded as global and local objects.(Cf. §1.1.) The above lemma shows that certain properties of global data are derived from those of local data(by under--standing the words local and global as explained in the in--troduction.).

<u>Proof of Lemma 1.1</u>. We first show the following facts under the assumptions in Lemma 1.1:

$(1.10)'$ Let $m \in Z^+$. Then for any element X_m in S_C of the form: $X_m = T_m(S_{\lambda_1}, \dots, \lambda_t)$,

$(1.10)'_1$ $H^*(C_0(X_m))$ is a finitely generated R-module,

$(1.10)'_2$ $\alpha^*(X_m)$ is bijective.

The above two facts will be shown by the induction on m: If m = 1, then the disjoint condition for $C_i(i=0,1,2)$ suffic -es to assure $(1.10)'_{1,2}$. We assume that $(1.10)'_{1,2}$ are true for m m. Then the Mayer-Vietoris condition for $C_i(i=0,1,2)$ as well as the five lemma suffices to assure $(1.10)'_{1,2}$ for m+1.

To show the original assertion $(1.10)_{1,2}$, let $S_C(m)$ denote the subset of S_C consisting of those elements $X \in S_C$ of the form $X = T_m$ with a suitable $T \in S_C$. We will show the assertions $(1.10)_{1,2}$ for $S_C(m)$ inductively on m. If m = 1, then the identity and the disjoint conditions suffice to assure $(1.10)_{1,2}$. Assume that $(1.10)_{1,2}$ are true for $\tilde{m} \leqq m$. Then the Mayer-Vietoris condition and the five lemma suffice to assure the assertions in $(1.10)_{1,2}$ for any $X \in S_C(m+1)$. This finishes the proof of Lemma 1.1.

2. The singular C.C.I. and the C^∞-de Rham C.C.I.

In this section we first define the notion of C^∞-thicke--ning for a prestratified space. We then define two C.C.I.'s ,the singular and the C^∞-de Rhem C.C.I.'s, for a prestratifi--ed space endowed with a C^∞-thickening.

In this section all the strata in question are C^∞- man--fold and for a point P_λ on a startum S_λ in question, $\dim_{P_\lambda} S_\lambda$ is the dimension of S_λ at P_λ (as a C^∞-manifold). We use symbols $S = \{S_\lambda, S_\lambda \in S\}$'s expressing collections of C^∞-manifolds for prestratifications.

Arguments in this section require finiteness conditions on prestratifications in question. We assume that every pres--tratification S in question is a finite set.

§2.1. C^∞-thickenings of prestratified spaces. Let M be a C^∞-manifold and V a subset of M. We call an open neighbour--hood N(V) in M a C^∞-thickening of V in M* if

(2.1) the natural isomorphism $i^*: H^*(N(V),R) \to H^*(V,R)$ induced from the inclusion $i: V \hookrightarrow N(V)$ is bijective.

When there is no fear of confusions, we call a C^∞-thick--ening N(V) of V in M a C^∞-thickening N(V) of V.

Now let (V,S) be a prestratified space,where V is a sub--set of a C^∞-manifold M. Moreover, let $N = \{N(S_\lambda), S_\lambda \in S\}$ be

(*) In this section we discuss explusively cohomological aspects of neighbourhoods. For homotopical aspects of neighb--ourhoods of strata, see the general theory in R.Thom[ ]. See also [ ],[ ] for homotopical versions in our contexts.

a collection of open neighbourhoods $N(S_\lambda)$'s of S_λ's in M. For an element T in the C-set S_C of S (cf. § 1) we write the union $\bigcup N(S_\lambda)$, $S_\lambda \in T$, as N(T). Also for an element $S_{\lambda_1},\dots,\lambda_t \in S_O$ and an element $T_m(S_{\lambda_1},\dots,\lambda_t) \in S_{CO}$, we write $\bigcap N(S_{\lambda_j})$ and $\{\bigcap N(S_{\lambda_j})\} \cap N(T_m)$ as respectively $N(S_{\lambda_1},\dots,\lambda_t)$ and $N(T_m(S_{\lambda_1},\dots,\lambda_t))$. Furthermore, let $T \in S_C$ and N' a subset of M. We write the intersection $N' \cap T$ as N'_T. We call the collection $\{N(S_\lambda)_T; S_\lambda \in T\}$ the contraction of **N** to T and write it as N_T. We use the above symbols $N(T), N(S_{\lambda_1},\dots,\lambda_t), N(T_m(S_{\lambda_1},\dots,\lambda_t)),\dots$ throughout this paper.

After the above preparations we introduce the notion of C^∞-thickening for a prestratified space ~~as follows:~~ in the following manner:

Definition 2.1. Let M be a C^∞-manifold, and V a subset of M. Moreover, let $S = \{S_\lambda, \lambda \in \Lambda\}$ be a prestratification of V. A collection $N = \{N(S_\lambda), S_\lambda \in S\}$ of C^∞- thickenings of S_λ's in M will be called a C^∞-thickening of S in M if the following conditions are satisfied.

$(2.2)_1$ If $N(S_\lambda) \cap N(S_\mu) \neq \phi$, then $S_\lambda \prec S_\mu$ or $S_\mu \prec S_\lambda$.

$(2.2)_2$ If $N(S_\mu) \cap S_\lambda \neq \phi$, then $S_\mu \prec S_\lambda$.

$(2.2)_3$ For any $T \in S_C$ and for any $X \in T_O$, the natural homomorphism $i^*(X)$: $H^*(N(X);R) \to H^*(N(X)_T;R)$ induced from the inclusion: $i(X)$: $N(X)_T \hookrightarrow N(X)$ is bijective.

$(2.2)_4$ For any $X \in S_C, S_O$ or S_{CO}, $N(X)$ is a paracompact C^∞-manifold.

If there is no fear of confusions, we call a C^∞-thick-ening N of S in M a C^∞-thickening N of S.

§2.2. The singular C.C.I. and the C^∞-de Rham C.C.I.

Let M be a C^∞-manifold and V a subset of M. Moreover, let $S = \{S_\lambda, \lambda\in\Lambda\}$ be a prestratification of V and $\mathbf{N} = \{N(S_\lambda); \lambda\in\Lambda\}$ a C^∞-thickening of S in M. We shall define two C.C.I.'s, called the singular and the C^∞-de Rham C.C.I.'s, for the pair (S,N) in the following manner:

(i) Let X be an element in one of three sets: S_C, S_O or S_{CO}. We denote by $\Delta(N(X))$ and $\Omega(N(X))$ respectively the co-chain complexes of the singular cochains over $\mathbf{R}$ and the C^∞-differential forms in N(X). We write the collections $\{\Delta(N(X)); X\in S_C, S_O$ or $S_{CO}\}$ and $\{\Omega(N(X)), X\in S_C, S_O$ and $S_{CO}\}$ respectively as $\Delta(S,N)$ and $\Omega(S,N)$. We call these collections respectively the singular and the C^∞-de Rham collections attached to the pair (S,N).

(ii) Here we define two collections, denoted by $K(\Delta(S,N))$, $K(\Omega(S,N))$, of homomorphisms for the pair (S,N) so that $(\Delta(S,N), K(\Delta(S,N)))$ as well as $(\Omega(S,N), K(\Omega(S,N)))$ become C.C.I.'s attached to S: We define homomorphisms K(Y) for $Y\in S_D$, according to the nature of $Y\in S_D$. To simplify notations, we use the symbol $\square$ for both Δ and Ω in $(ii)_1 \sim (ii)_3$ below.

$(ii)_1$ Let $S_\lambda \in S$. Then it is clear that $\square(N(\{S_\lambda\})) = \square(N(S_\lambda))$. (In the right side of this equality, we should regard S_λ as an element in S_0.) We define $K(\square;S_\lambda)$ to be the identity map $i(\square;S_\lambda)$ $: \square(N(\{S_\lambda\})) \longrightarrow \square(N(S_\lambda))$.

$(ii)_{2,1}$ Let T_1, T_2 be elements in S_C such that $T_1 \vee T_2$. By (2.2), $N(T_1 \cup T_2)$ coincides with the disjoint union: $N(T_1) \cup N(T_2)$ Therefore, the inclusion $i(T_1,T_2) : \square(N(T_1) \uplus N(T_2)) \to \square N(T_1 \cup T_2)$ induces the natural isomorphism

$$i(\square;T_1,T_2): \quad 0 \to \square(N(T_1 \cup T_2)) \to \square(N(T_1)) \underset{\square}{\oplus} (N(T_2)) \to 0 .$$

We write this isomorphism as $K(\square;T_1,T_2)$.

$(ii)_{2,2}$ Let $X_1 = T_1(S_{\lambda_1},\ldots,\lambda_t) \in S_{CO}$. We write X_1 explicitly as $X_1 = \{S_{\lambda^1_{t+1}}\}$. Then by (2.2), $N(X_1)$ coincides with the disjoint union: $\cup N(S_{\lambda^1_{t+1}})$. The inclusion $i(X_1): \cup N(S_{\lambda^1_{t+1}}) \hookrightarrow N(X_1)$ induces naturally the isomorphism

$$i(\square;X_1) \quad 0 \to (N(X_1)) \to \oplus \square(N(S_{\lambda^1_{t+1}})) \to 0.$$

We write this isomorphism $i(\square;X_1)$ as $K(\square;X_1)$.

$(ii)_{3,1}$ Let $(T,m) \in S_C \times Z^+$. We write $T_{m+1} - T_m$ explicitly as $\{S_{\lambda^{m+1}}\}$. Then we have the following relations:

$$N(T_m) \ \begin{matrix}\cup\\ \cap\end{matrix} \ \{\cup N(S_{\lambda^{m+1}})\} = \begin{cases} N(T_{m+1}) \\ \cup N(T_m(S_{\lambda^{m+1}})) \end{cases}$$

From the above relations, we obtain the short exact sequence (Mayer-Vietoris seqience):

$$0 \to \square(N(T_{m+1})) \xrightarrow{h_1(\square;T,m)} \square(N(T_m)) \oplus \{\oplus \square(N(S_{\lambda^{m+1}}))\}$$

$$\xrightarrow{h_2(\square;T,m)} \oplus \square(N(T_m(S_{\lambda^{m+1}}))) \to 0.$$

We write $h_i(\square;T,m)$ as $K_i(\square;T,m)$ $(i=1,2)$.

$(ii)_{3.2}$ Finally let $X_m = T_m(S_{\lambda_1},\dots,\lambda_t) \in S_{CO}$. We write $T_{m+1}(S_{\lambda_1},\dots,\lambda_t)$ as X_{m+1} and $X_{m+1} - X_m$ as $\{S_{\lambda^{m+1}_{t+1}}\}$. Then by an argument pararell to that in $(ii)_{3.1}$, we obtain the following exact sequence:

$$0 \longrightarrow \square(N(X_{m+1})) \xrightarrow{h_1(\square;X_m)} \square(N(X_m)) \oplus \{\oplus \square(N(S_{\lambda_1},\dots,\lambda^{m+1}_{t+1}))\}$$
$$\xrightarrow{h_2(\square;X_m)} \oplus \square(N(T_m(S_{\lambda_1},\dots,\lambda^{m+1}_{t+1}))) \longrightarrow 0.$$

We write $h_i(\square;X_m)$ as $K_i(\square;X_m)$ $(i=1,2)$.

We write the collection of homomorphisms constructed in $(ii)_1 \sim (ii)_3$ as $K(\square(S,N))$ and $K(\Omega(S,N))$ according to whether $\square = \Delta$ or Ω. Also to make clear the dependence of homomorphis--ms $K(\square;Y)$'s on the pair (S,N), we will write $K(\Delta;Y)$ and $K(\Omega;Y)$ as $K(\Delta(S,N);Y)$ and $K(\Omega(S,N);Y)$.

(iii) From the arguments in (i) and (ii), it is clear that the pairs $\Delta(S,N) = (\Delta(S,N),K(\Delta(S,N)))$ and $\Omega(S,N) = (\Omega(S,N),K(\Omega(S,N)))$ are C.C.I.'s attached to the prestratificati--on S. We call these C.C.I.'s respectively <u>the singular and the C^∞-de Rham C.C.I.'s attached to the pair (S,N)</u>.

(iv) Let $T \in S_C$, and let N_T be the cotraction of N to T. We shall define a C.C.I., called <u>the contraction of $\Delta(S,N)$ to T</u> in the following manner: (a) Let $X' \in T_C, T_O$ or T_{CO}. Then

we write the cochain complex $\Delta(N(X')_T)$ of the singular coch-
-ains (over R) as $\Delta(N_T;X)$. We write the collection $\{\Delta(N_T;X');$ $X'\in S_C,S_0$ or $S_{C0}\}$ as $\Delta(N_T)$ and call it the contraction of $\Delta(S,N)$ to T. (b) Corresponding to considerations in $(ii)_1\sim(ii)_{3,2}$ take a stratum $S'_\lambda\in T'$, a pair $(T'_1,T'_2)\in T_C\times T_C$ such that $T'_1\vee T'_2$ $,\dots, X'_m=T'_m(S'_{\lambda_1},\dots,\lambda_t)\in T_{C0},\dots$. Also corresponding to nei-
-ghbourhoods $N(S_\lambda),N(T'_1\cup T'_2),\dots N(X_m),\dots$ in the considerations in $(ii)_1,(ii)_{2,1},\dots(ii)_{3,2}$ we take intersections $N(S'_\lambda)_T$, $N(T'_1\cup T'_2)_T,\dots,N(X'_m)_T,\dots$ We then restrict homomorphisms $K(\Delta(S,N),S'_\lambda),K(\Delta(S,N),T'_1,T'_2),\dots,K(\Delta(S,N),X'_m),\dots$ to $\Delta(N_T;S_\lambda)$ $\Delta(N_T;T'_1\vee T'_2),\dots,\Delta(N_T;X'_m),\dots$ We write the resulting homomor-
-phisms as $K(\Delta(N_T);S'_\lambda),K(\Delta(N_T);T'_1,T'_2),\dots,K(\Delta(N_T);X'_m),\dots$

We write the collection of these homomorphisms $K(\Delta(N_T);Y)$'s, $Y=S'_\lambda,(T'_1,T'_2),\dots,$ as $K(\Delta(N_T))$. Then it is clear that the pair $\Delta(N_T)=\{\Delta(N_T),K(\Delta(N_T))\}$ is a C.C.I. attached to the pre-
-stratification T of $|T|$, We call this C.C.I. the contraction of $\Delta(S,N)$ to T.

§ 2.3. Equivalences between the singular and the C^∞-de Rh-
-am C.C.I.'s,etc.

Let M be a C^∞-manifold and V a subset of M. Moreover, let S be a prestratification of V and N a C^∞-thickening of V in M. Let $\Delta(S,N)$ and $\mathfrak{R}(S,N)$ denote respectively the singular and the C^∞-de Rham C.C.I.'s. Now take an element $X\in S_C,S_0$ or

S_{CO}. Then, by the theorem of de Rham([ ]), there exists a canonical isomorphism

$$d^*:\ H^*(\Omega(N(X))) \longrightarrow H^*(\Delta(N(X))).$$

We write the collection $\{d^*(X), X \in S_C, S_O \text{ or } S_{CO}\}$ as d^*. Then it is clear that d^* commutes with $K^*(\Delta(S,N))$ and $K^*(\Omega(S,N))$, where K^*'s as above are collections of homomorphisms $K^*(Y)$'s induced from $K(Y)$'s, $Y \in S_D$. Therefore we obtain the following

Proposition 2.1. Let the notations be as in the beginning of §2.3. Then the singular C.C.I. $\Delta(S,N)$ and the C^∞-de Rham C.C.I. $\Omega(S,N)$ are equivalent (in a natural manner).

Next let $T \in S_C$. we denote by $\Delta(S,N)_T$ and $\Delta(N_T)$ respectively the restriction and the contraction of $\Delta(S,N)$ to T. Take an element $X' \in T_C, T_O$ or T_{CO}. Then the inclusion $i(X')$: $N(X')_T \hookrightarrow N(X')$ induces a homomorphism $i^*(X')$: $H^*(\Delta(S,N)_T;X) \longrightarrow H^*(\Delta(N_T;X))$ naturally. We write the collection $\{i^*(X'), X' \in T_C, T_O \text{ or } T_{CO}\}$ as I^*, It is clear that I^* commutes with collections $K^*(\Delta(S,N))_T$ and $K^*(\Delta(N_T))$ of homomorphisms of cohomology groups(obtained from K's). This fact, together with $(2.2)_3$, leads to the following

Proposition 2.2. Being notations as above, the restriction $\Delta(S,N)_T$ and the contraction $\Delta(N_T)$ of the singular C.C.I. $\Delta(S,N)$ to $T \in S_C$ are equivalent.

Now we conclude this section by the following

Lemma 2.1. Let M be a C^∞-manifold and V a subset of M. Moreover, let (S,N) be a pair consisting of a prestratification S of V and a C^∞-thickening of S in M. Then for any $T \in S_e$, we have a natural isomorphism :

$$(2.3) \qquad H^*(\Omega(S,N);T) \xrightarrow{\ \simeq\ } H^*(|T|;R).$$

Here $\Omega(S,N)$ is the C^∞-de Rham C.C.I. attached to (S,N).

Proof. Besides the C.C.I. $\Omega(S,N)$, let $\Delta(S,N)$ denote the singular C.C.I. attached to (S,N). Then, from Proposition 2.2. and Lemma 1.1, we have a natural isomorphism

$$H^*(\Delta(S,N)_T;T) \simeq H^*(\Delta(N_T;T)).$$

Clearly, $H^*(\Delta(S,N)_T;T) = H^*(\Delta(S,N);T))$ and $H^*(\Delta(N_T;T)) \simeq H^*(|T|;R)$. On the otherhand, we know from Proposition 2.1 that

$$H^*(\Omega(S,N);T) \simeq H^*(\Delta(S,N);T)).$$

From the above observations, (2.3) follows immeadiately.

~~Remarks to §1,2~~

§ 3. P.G.-adequate prestratification(*) and P.G.-adequate C^∞-de Rham C.C.I.

In this section we assume that each stratum in question is an equidimensional real manifold.

§ 3.1. Definitions. (1) Let $R^n(x)$ (resp. $C^n(z)$) be a real (resp. complex) euclidean space of dimension n. We assume that $R^n \subset C^n$ and that $\mathrm{Re} z_i = x_i; i=1,..,n$. For points $\tilde{P}_1=(z^1)$ and $\tilde{P}_2=(z^2)$ in C^n let $d(\tilde{P}_1, \tilde{P}_2)$ denote the 'natural' distance $\sum_{j=1}^n \{|z_j^1 - z_j^2|^2\}^{1/2}$. Moreover, for a point $\tilde{P} \in C^n$ and a subset $\tilde{\Sigma} \subset C^n$, define $d(\tilde{P},\tilde{\Sigma})$ to be $\inf_{\tilde{Q}\in\tilde{\Sigma}} d(\tilde{P},\tilde{Q})$. For a point $\tilde{P}$ and $r>0$, $\tilde{\Delta}(\tilde{P}:r)$ stands for a disc with the center $\tilde{P}$ and the radius r. If $P \in R^n$, then $\Delta(P;r) = \tilde{\Delta}(P;r) \cap R^n$. For two subsets $\tilde{\Sigma}_1$, $\tilde{\Sigma}_2 \neq \phi$ in C^n and a couple $(\delta) = (\delta_1, \delta_2)$ of positive numbers, define an open neighbourhood $\tilde{N}_\delta(\tilde{\Sigma}_1, \tilde{\Sigma}_2)$ of $\tilde{\Sigma}_1 - \tilde{\Sigma}_2$ by

$$\tilde{N}_\delta(\tilde{\Sigma}_1, \tilde{\Sigma}_2) = \bigcup_{P_1} \tilde{\Delta}(P_1 ; \delta_1 \cdot d(P_1, \tilde{\Sigma}_2)^{\delta_2}), \text{where } P_1 \in \tilde{\Sigma}_1 - \tilde{\Sigma}_2 .$$

If $\Sigma_1, \Sigma_2 \neq \phi \subset R^n$, then define $N_\delta(\Sigma_1, \Sigma_2) = \tilde{N}_\delta(\Sigma_1, \Sigma_2) \cap R^n$. In the above definitions we assumed that $\Sigma_2 \neq \phi$. In the later arguements it is convenient to use similar symbols for the case where $\Sigma_2 = \phi$. Let (δ) be a couple of the form: $(\delta) = (\delta_1, 0)$. Then we define $\tilde{N}_\delta(\tilde{\Sigma}_1) = \tilde{N}(\tilde{\Sigma}_1, \phi)$ by $\tilde{N}(\tilde{\Sigma}_1) = \bigcup_{P\in\Sigma_1} \Delta(P; \delta_1)$.

(*) P.G. = polynomial growth.

Remark. In the sequel of the present paper the use of the symbols $N_\delta(\tilde{\Sigma}_1,\tilde{\Sigma}_2)$ is do strictly in the above senses: To be sure, if $\tilde{\Sigma}_2 \neq \phi$, a couple (δ) is always a couple (δ_1,δ_2) of positive numbers. If $\tilde{\Sigma}_2=\phi$, a couple (δ) is always of the form $(\delta_1,0)$.

Let $\tilde{U}$ be a bounded set in C^n and $\tilde{\Sigma}_1$ a subset of C^n such that $\tilde{\Sigma}_1 \cap \tilde{U} \neq \phi$. Moreover, let $\tilde{\Sigma}_2$ be a subset of C^n such that $\tilde{\Sigma}_1 \cap \tilde{\Sigma}_2 \cap \tilde{U} \neq \phi$. The set $\tilde{\Sigma}_2$ <u>intersects (d)-regularly</u> with $\tilde{\Sigma}_1$ in $\tilde{U}$ if

$$d(\tilde{P}_1,\tilde{\Sigma}_2) \sim d(\tilde{P}_1, \tilde{\Sigma}_1 \cap \tilde{\Sigma}_2) \text{ for any } \tilde{P}_1 \in \tilde{\Sigma}_1 \cap U .$$

Remark. The notion of 'intersect (d)-regularly' is inspired by that of 'regularly suited' in Marglange [ ]. The former is essentially same as the later except modifications of technical natures.

Let $\tilde{U}$ be a bounded set in C^n. We then define the radius $r(\tilde{U})$ of $\tilde{U}$ by $r(\tilde{U}) = \sup_{\tilde{P},\tilde{Q}\in U} d(\tilde{P},\tilde{Q})$. For a bounded set $\tilde{U}$ in C^n, a subset $\tilde{U}'$ of C^n is a <u>(d)-envelop of $\tilde{U}$</u> if the following condion $(\bigstar)$ is valid.

$(\bigstar)$ For any $\tilde{P}\in\tilde{U}$, $\tilde{\Delta}(\tilde{P}; kr(\tilde{U}))\subset \tilde{U}'$, where k is in R such that $k \geqq 2$.

If $\tilde{U}'$ is a (d)-envelop of $\tilde{U}$, then the following $(\bigstar)'$ is obvious.

$(\star)'$ For a subset $\tilde{\Sigma}\subset C^n$ satisfying $\tilde{\Sigma}\cap\tilde{U}\neq\phi$, $d(P,\tilde{\Sigma}\cap\tilde{U}')=d(P,\tilde{\Sigma})$ for any $\tilde{P}\in\tilde{U}$.

Henceforth our arguements will be done in R^n. Let U be a bounded domain and U' a (d)-envelop of U.(*) Moreover, let U'' be a domain in R^n containing U'. Furthermore, let f be a real analytic function defined in U' and V the zero locus of f in U'. Then the following is also obvious.

$(\star)''$ If $V\cap U\neq\phi$, then $f(P)\sim d(P,V)=d(P,V\cap U')$ for any $P\in U$.

Remark. Let U,U' and U'', f be as in $(\star)''$. Choose couples (a), (a') such that $f(P)\underset{(a):(a'')}{\sim}d(P,V\cap U')$. Then, for any U''' such that $U'\subset U'''\subset U''$, we have $f(P)\underset{(a)\ (a')}{\sim}d(P,V\ U''')$.

(2) A bounded domain U satisfying

(3.1) $r(U)<1/2$,

is called to satisfy (d)-regularization condition. Let U be such a domain and V a closed set in U. For a pair (U,V), a pair $(U'.V')$ of a bounded domain $U'\subset R^n$ and a closed set V' in U' is a (d)-envelop of (U,V) if

(3.2) U' is a (d)-envelop of U and $V'\cap U=V$.

Let $(U,V),(U',V')$ be as above, and let (S,S') be a pair of prestratifications of V and V' respectively. If S is a restriction of S' to V, then the triple (U',V',S')

(*) If a subset $U\subset R^n$, then a subset $U'\subset R^n$ will be called a (d)-envelop of U provided $U'=\tilde{U}'\cap R^n$. Here $\tilde{U}'\subset\mathbb{C}^n$ is a (d)- envelop of U in the originsl sense.

is called a (d)-envelop of (U,V,S). For the above (S,S') we say that (S,S') satisfies $(\tilde{\delta})$-separation condition if the following is true.

(3.3) With each $S \in S$ associated is a couple $(\gamma) = \gamma(S)$ such that if $N_{\gamma}(S_{\lambda'}, \mathrm{fron}(S_{\lambda'})) \cap N(S_{\lambda''}, \mathrm{fron}(S_{\lambda''})) \neq \phi$, then $S_{\lambda'} < S_{\lambda''}$ or $S_{\lambda''} < S_{\lambda'}$.

In (3.2) the rule in the use of a couple $(\tilde{\delta})$ is kept. (Cf. the firsr remark in § 3.1.). Moreover, (S,S') satisfies (d)-regular intersection condition if the following is true

(3.4) For any $S'_{\lambda'}$ $S'_{\lambda''} \in S'$ such that $S'_{\lambda'} \cap S'_{\lambda''} \cap U \neq \phi$, $S'_{\lambda'}$ (resp. $S'_{\lambda''}$) intersects $S'_{\lambda''}$ (resp. $S'_{\lambda'}$) regularly in U.

For a relation between the above two conditions, see the remark in the end of this section.

(3) Let (U,V) be as in the beginning of (2) and S a prestratification of V. Moreover, let (U',V',S') a (d)-envelop of (U,V,S). Let $F' = \{f'(S); S \in S\}$, $G' = \{g'(S);\ S \in S, \mathrm{fron}(S) \neq \phi\}$ and $H' = \{h'(T,S)\ S \in S,\ T \in S_c$ such that $S \in T\}$ of real analytic functions. Such families F', G, H will be families be called families of comparison functions (f.c.f) associated with (S,S') if the followings are true.

$(3.5)_1$ Functions $f'(S)$, $g'(S)$ and $h'(T,S)$'s are defined in U'. $N_{\gamma}(S', \mathrm{fron}(S'))$, where $\gamma = \delta(S)$ is a suitable couple and $S' \in S'$ such that $S' \cap U = S$.

$(3.5)_2$ For each $S \in \mathcal{S}'$, we have the following comparisons:

(i) $f'(S:Q) \sim d(Q,S)$ and $g'(S:Q) \sim d(Q, \mathrm{fron}(S))$ for any $Q \in N_{\tilde{\delta}}(S, \mathrm{fron}(S))$.

(ii) $h'(T,S:Q) \sim d(Q, T''^{+}(S))$ for any $Q \in N_{\tilde{\delta}}(S, \mathrm{fron}(S)) \cap N(T''^{+}(S)\ \overline{S})$. Here $\tilde{\delta} = \tilde{\delta}(S,T)$ is a suitable couple and $T''^{+}(S) = T'^{+}(S) - \{S';\ \dim S' \neq n\}$.

(4) Let (U,V) be as in the beginning of (2). A finite prestratification $\mathcal{S}$ of V is P.G.-adequate (polynomial growth adequate) if we can find the following data.

$(3.6)_1$ A triple $(U',V',\mathcal{S}')$ of a (d)-envelop of $(U,V,\mathcal{S})$ satisfying (δ)-separation condition (and so the regular intersection condition) (cf. the end of this section.)

$(3.6)_2$ Families F',G,H associated with $(\mathcal{S},\mathcal{S}')$.

When we emphasize $(U',V',\mathcal{S}')$ and (F',G',H') we say that $\mathcal{S}$ is P.G.-adequate with $(U',V',\mathcal{S}')$ and (F',G',H').

§ 3.2. Sets of C^∞- functions. Let (U,V) be as in (4), § 3.1. Moreover, let $\mathcal{S}$ be a P.G.-adequate prestratification of V with $(U',V',\mathcal{S}')$ and (F',G',H'). Fix a C^∞-function $\chi_0(t)$; $t \in R$, satisfying the conditions :(i) $0 \leqq \chi_0(t) \leqq 1$, (ii) $\chi_0(t) = 1$, if $t \leqq 1/2$ and $\chi_0(t) = 0$, if $t \geqq 1$ .. Now we define C^∞-functions of the following two types:

(i) Let $S \in \mathcal{S}$ such that $\mathrm{fron}(S) \neq \phi$. For a couple (a)

of positive numbers , define a C^∞- function $\chi_a(f',g':S)$ by

$$\chi_a(f',g':S) = \chi_0(\, a_1 \cdot f'^{a_2}/g'\,)\ .$$

This function χ_a is defined in $U' -$ fron(S') ,where $S' \in \mathcal{S}'$ such that $S' \cap U = S$.

(ii) Let (S,T) be a pair of $S \in \mathcal{S}$ and $T \in \mathcal{S}_C$. For a couple $(a) = (a_1,a_2)$, define a C^∞-function $\chi_a(f',h': S,T)$ by

$$\chi_a(f',h':S,T) = \chi_0(\, a_1 h'^{a_2}/f'\,)\ .$$

The function $\chi_a(f',h')$ is defined in$(U' - S) \cap N_\delta(S:\text{fron}(S))$.

The following properties of functions $\chi_a(f',g')$ and $\chi_a(f',h')$ will be utilized in the later arguements:

(i)' Let S be as in (i). Given a couple (δ), we find a couple (a) and a couple $(\delta') = (\delta'(a))$ such that

(3.7)$_1$ $\chi_a(f',g')$ =0 outside $N_\delta(S,\text{fron}(S))$,and $\chi_a(f',g')$ = 1 in $N_{\delta'}(S,\text{fron}(S))$.

(ii)' Let S,T be as in (ii). Given a couple (δ), we find a couple (a) and a couple $(\delta') = (\delta'(a))$ such that

(3.7)$_2$ $\chi_a(f',h')$ =0 outside $N_\delta(T''^+(S),\bar{S}) \cap N(S,\text{fron}(S))$, and = 1 in $N_{\delta'}(T''^+(S),\bar{S}) \cap N(S,\text{fron}(S))$.

§ 3.3. Assignments $\widehat{\Omega}(S,N)$ and $\widehat{\Omega}'_{p.g.}(S,N)$. (1) Let (U, V) be as in the beginning of (2), § 3.1. Moreover, let S be a P.G.-adequate prestratification of V. A C^∞-thickening N of a P.G.-adequate prestratification S is always assumed to satisfy the following condition in addition to (2.1), (2.2).

($\star$) For each $S \in S$, $N_{\delta'}(S,\mathrm{fron}(S)) \subset N(S) \subset N_\delta(S,\mathrm{fron}(S))$ with suitable couples $(\delta) = (\delta(S))$, $(\delta') = (\delta'(S))$.

Let $N = \{N_j\ ; j = 1,2,\dots\}$ be a system of C^∞-thickenings of a P.G.-adequate S. We say that N is a direct system of C^∞-thickenings (with respect to inclusions) of S, if, in addition to the conditions $N_{j'} < N_j$; $j' < j$, the following condition is valid.

(3.8) For any $S \in S$ and any couple (δ), $N_j(S) \subset N_\delta(S,\mathrm{fron}(S))$ with a suitable j.

(3.8)' $\bigcap_j N_j(S) = S$ for each S.

For a direct system $N = \{N_j\}$ of a P.G.-adequate S, we define, for each j, assignmemts $C'(\Omega_{p.g.j}(S,N))$, $E(\Omega_{p.g.j}(S,N))$ and $E(\Omega_{p.g.j}(S,N))$ by the following formulas.

$$(3.9)_1 \quad C'(\Omega_{p.g.j}(N,S))\ ;\ \tau \in S_C \longrightarrow \Omega_{p.g.}(N_j(\tau);\mathrm{fron}(\tau)),$$

$$(3.9)_2 \quad C'(\Omega_{p.g.j}(N,S)) : \tau \in S_0 \longrightarrow \Omega_{p.g.}(N_j(\tau),\mathrm{fron}(\tau)),$$

$$(3.9)_3 \quad C'(\Omega_{p.g.j}(S,N)) : \tau \in S_{0C} \longrightarrow \Omega_{p.g.}(N_j(\tau),\mathrm{fron}(\tau)).$$

Obviously $\Omega_{p.g.j}$'s in ths right side of $(3.9)_{1,2,3}$ are graded differential rings with the differential operator d. Also note that the above $\Omega_{p.g.j}(\mathcal{U},\mathrm{fron}(\mathcal{U}))$ are determined by N_j and by $\mathcal{U}\in S_C$, $\mathcal{U}\in S_O$, $\mathcal{U}\in S_{OC}$ uniquely. We, therefore, denote them by $\Omega_{p.g.j}(\mathcal{U})$; $\mathcal{U}\in S_C$, S_o or S_{OC}. For an element $\mathcal{U}\in S_C, S_O$ or S_{OC}, let $\rho_{j'j}(\mathcal{U})$; $j'<j$, denote the induced homomorphism $\rho_{j'j}: \Omega_j(\mathcal{U}) \longrightarrow \Omega_{j'}(\mathcal{U})$ * from the inclusion $i_{j'j}$; $N_{j'}(\mathcal{U}) \hookrightarrow N_j(\mathcal{U})$. Then, for any $\mathcal{U}$, $\rho_{j'j}(\mathcal{U})(\Omega_{p.g.j}(\mathcal{U})) \hookrightarrow \Omega_{p.g.j'}(\mathcal{U})$. The restriction of $\rho_{j'j}(\mathcal{U})$ to $\Omega_{p.g.j}(\mathcal{U})$ will be denoted by $\rho_{p.g.j'j}$. Now we have the following direct systems of graded differential rings of two types: $\{\Omega_j(\mathcal{U}), \rho_{j'j}(\mathcal{U})\}_{\mathcal{U}}$ $\{\Omega_{p.g.j}(\mathcal{U}), \rho_{p.g.j'j}\}_{\mathcal{U}}$. We let, for any $\mathcal{U}$, $\hat{\Omega}(S,N:\mathcal{U})$ and $\hat{\Omega}_{p.g.}(S,N;\mathcal{U})$ be the dúrect limits $\varinjlim \Omega(N_j(\mathcal{U}))$ and $\varinjlim \Omega_{p.g.j}(\mathcal{U})$ respectively. The differntial operator d commutes with $\rho_{j'j}$'s and $\rho_{p.g.j'j}$'s, and $\hat{\Omega}=\hat{\Omega}(S,N:\mathcal{U})$, $\hat{\Omega}_{p.g.}(S,N,\mathcal{U})$ are graded differential rings with the operat-or d in an obvious manner. Define assignments $C'(\hat{\Omega}(S,N))$, $E(\hat{\Omega}(S,N))$, $E'(\hat{\Omega}(S,N))$ (resp. $C'(\hat{\Omega}_{p.g.}(S,N))$, $E(\hat{\Omega}_{p.g.}(S,N))$, $E'(\hat{\Omega}_{p.g.}(S,N))$ by the following formulas.

$$(3.10)_1 \begin{cases} C'(\hat{\Omega}(S,N)) \; ; \quad T\in S_C \longrightarrow \hat{\Omega}(S,N:T), \\ C'(\hat{\Omega}_{p.g.}(S,N)) : \quad T\in S_C \longrightarrow \;\; {}_{p.g.}(S,N:T). \end{cases}$$

(*) $\Omega_j(U) = \Omega(N_j(U))$.

$$(3.10)_2 \begin{cases} E(\widehat{\Omega}(S,N)) : \tau \in S_0 \longrightarrow \widehat{\Omega}(S,N:\tau), \\ E(\widehat{\Omega}_{p.g.}(S,N)) : \tau \in S_0 \longrightarrow \widehat{\Omega}_{p.g.}(S,N:\tau). \end{cases}$$

$$(3.10)_3 \begin{cases} E'(\widehat{\Omega}(S,N)) : \tau \in S_{0C} \longrightarrow \widehat{\Omega}(S,N:\tau), \\ E'(\widehat{\Omega}_{p.g.}(S,N)) : \tau \in S_{0C} \longrightarrow \widehat{\Omega}_{p.g.}(S,N:\tau). \end{cases}$$

Collections $C'(\widehat{\Omega}(S,N))$, $E(\widehat{\Omega}(S,N))$, $E'(\widehat{\Omega}(S,N))$ and $C'(\widehat{\Omega}_{p.g.}(S,N))$, $E(\widehat{\Omega}_{p.g.}(S,N))$, $E'(\widehat{\Omega}_{p.g.}(S,N))$ are denoted by $\widehat{\Omega}(S,N)$ and $\widehat{\Omega}_{p.g.}(S,N)$ respectively.

(2) Collections $K(\widehat{\Omega}(S,N))$ and $K(\widehat{\Omega}_{p.g.}(S,N))$. Let $\{\Omega_j(\tau), \rho_{j'j}(\tau)\}_\tau$ and $\{\Omega_{p.g.j}(\tau), \rho_{p.g.j'j}(\tau)\}_\tau$ be the direct systems of graded differential rings defined in (1) § 3.3. . In this part § 3.3.(2), we write $\rho_{j'j}(\tau)$ as $\alpha_{j'j}(\tau)$, $\beta_{j'j}(\tau)$ or $\beta'_{j'j}(\tau)$ according to whether $\tau \in S_C$, S_0 or S_{0C}. We write collections $\{\alpha_{j'j}(\tau)\ ;\tau \in S_C\}$, $\{\beta_{j'j}(\tau)\ ;\tau \in S_0\}$ and $\{\beta'_{j'j}(\tau)\ ;\tau \in S_{0C}\}$ as $\alpha_{j'j}$, $\beta_{j'j}$ and $\beta'_{j'j}$ respectively. Moreover, let $A_{j'j}$ denote the collection $\{\alpha_{j'j}, \beta_{j'j}, \beta'_{j'j}\}$.

Then it is easy to see that $A_{j'j}$ commutes with $K_j = K(\Omega(S,N_j))$ and $K_j' = K(\Omega(S,N_j))$; $1 \leq j \leq j'$. From the commutativity of $A_{j'j}$, with K_j and K_j' we obtain the following direct systems of homomorphisms: (i) $\{i_j(S); S \in S\}$, $(\text{ii})_1$ $\{i_j(T_1,T_2)\ ;\ T_1,T_2 \in S_C\}$, $(\text{ii})_2$ $\{i_j(\tau)$ for any $\tau = T_1^+(S_{\lambda_1},\ldots,\lambda_t)\}$, $(\text{iii})_1$ $\{I_{k,j}(T_m^+)\ ;\ T \in S_C, k=1,2\}$, and $(\text{iii})_2$ $\{I_{k,j}(\tau)\ ;\tau \in S_{0C}, k=1,2\}$.

In the above T_1 and T_2 are assumed to be independent.

From the above direct systems of homomorphisms we have the following direct limits: $(i)'\{\hat{i}(S); S \in S\}$, $(ii)'_1\{\hat{i}(T_1, T_2); T_1, T_2 \in S_C\}$ where $T_1 \vee T_2$, $(ii)_2\{\hat{i}(U)$; for any $U = T_1^+(S_{\lambda_1}, \ldots, {}_{\lambda_t})\}$, $(iii)_1\{\hat{I}_{kj}(T_m^+); T \in S_C, k=1,2\}$, and $(iii)_2\{\hat{I}_{kj}(U); U \in S_{OC}\}$. In the above $\hat{i}(S) = \varinjlim i_j(S), \ldots,$. Let $K(\hat{\Omega}(S,N))$ be the collection of all the above homomorphisms. Then $\hat{\Omega}(S,N)$ becomes a cochain complex with $K(\hat{\Omega}(S,N))$ (attached to S) in an obvious manner. Next recall that, for each $j \in Z'^+$, $\Omega_{p.g.j}(S,N) \lhd \Omega(S,N_j)$. Now we show the following

Proposition 太字 3.1. <u>For each j, the collection $K_j = K(\Omega(S,N_j))$ preserves $\Omega_{p.g.j}(S,N)$.</u>

Proof. We will show the following relations for maps in question.

$(3.11)_1 \quad 0 \longrightarrow C'_{p.g.j}(S) \xrightarrow{i_j(S)} E_{p.g.j}(S) \longrightarrow 0$.

$(3.11)_2 \quad 0 \longrightarrow C'_{p.g.j}(T_1T_2) \xrightarrow{i_j(T_1,T_2)} C'_{p.g.j}(T_1) \oplus$

$C'_{p.g.j}(T_2) \longrightarrow 0$.

$(3.11)_3 \quad 0 \rightarrow E_{p.g.j}(U) \xrightarrow{i_j(U)} \oplus_{\lambda_{t+1}} E_{p.g.j}(S_{\lambda_1,\ldots,\lambda_{t+1}}) \rightarrow 0$;

$U = T_1^+(S_{\lambda_1,\ldots,\lambda_{t+1}})$.

$(3.11)_4 \quad 0 \rightarrow C'_{p.g.j}(T_{m+1}) \xrightarrow{I_{1j}(T_m^+)} C'_{p.g.j}(T_m^+) \oplus \{\oplus_{\lambda^{m+1}}$

$C'_{p.g.j}(S_{\lambda^{m+1}})\}$,

$I_{2j}(T_m^+) : C'_{p.g.j}(T_m^+) \oplus \{\oplus_{\lambda^{m+1}} C'_{p.g.j}(S_{\lambda^{m+1}})\} \longrightarrow$

$\oplus_{\lambda^{m+1}} E'_{p.g.j}(T_m^+(S_{\lambda^{m+1}}))$.

$(3.11)_5 \quad 0 \rightarrow E'_{p.g.j}(U_{m+1}) \xrightarrow{I_{1j}(U_m)} E'_{p.g.j}(U_m) +$

$\{\oplus_{\lambda_{t+1}^{m+1}} E_{p.g.j}(S_{\lambda_1,\ldots,\lambda_{t+1}^{m+1}})\}$; $U_m = T_m^+(S_{\lambda_1,\ldots,\lambda_t})$,

$(\ =0,1),\ U_{m+1} = T_{m+1}^+(S_{1,\ldots,t})$.

$I_{2j}(U_m) : E'_{p.g.j}(U_m) \oplus \{\oplus_{\lambda_{t+1}^{m+1}} E_{p.g.j}(S_{\lambda_1,\ldots,\lambda_{t+1}^{m+1}})\}$

$\longrightarrow \{\oplus_{\lambda_{t+1}^{m+1}} E_{p.g.j}(S_{\lambda_1,\ldots,\lambda_{t+1}^{m+1}})$.

In the aboves $U_m = T_m^+(S_{\lambda_1},\ldots,S_{\lambda_t})$, $U_{m+1} = T_{m+1}^+(S_{\lambda_1},\ldots,S_{\lambda_t})$.

In the aboves $S, \ldots, \hat{T}_m^+(S_{\lambda_1}, \ldots, \lambda_t)$ have the same meanings as in (1.5), (1.6) , (1.7). We show the above 'obvious' relations by checking each case. At first $(3.11)_1$ is trivial from the definitions of $C'_{p.g.j}$ and $E'_{p.g.j}$. Next $(5.7)_{2,3}$ are immeadiate from the following remaks : (a) The independence of T_1 and T_2 implies the relation : $d(Q_i, \mathrm{fron}(T_i)) \sim d(Q_i, \mathrm{fron}(T_1 \cup T_2))$ for each $Q_i \in N_j(T_i)$, $i = 1,2$. (a') For $T_1^+(S_{\lambda_1}, \ldots, \lambda_t) = \{S_{\lambda_{t+1}}\}$, strata $S_{\lambda_{t+1}}$'s are independent each other. Then $d(Q, \mathrm{fron}(S_{\lambda_{t+1}})) \sim d(Q, \mathrm{fron}(T_1^+(S_{\lambda_1}, \ldots, \lambda_t)))$ for each $Q \in N_j(S_{\lambda_1}, \ldots, \lambda_{t+1})$. Concerning $(3.11)_{3,4}$ first note obvious relations : $\mathrm{fron}(T_{m+1}^+) \subset \mathrm{fron}(T_m)$, $\mathrm{fron}(U_{m+1}) \subset \mathrm{fron}(U_m)$, where $T_{m+1}^+, \ldots, U_m$ are as in $(3.11)_{3,4}$. Combining the above remaks with the facts (a_1), (a'_1),

(a_1) $d(Q, \mathrm{fron}(S_{\lambda_{m+1}})) \sim d(Q, \mathrm{fron}(T_{m+1}^+))$ for $Q \in N_j(S_{\lambda_{m+1}}, \mathrm{fron}(S_{\lambda_{m+1}}))$,

(a'_1) $d(Q, \mathrm{fron}(S_{\lambda_{t+1}^{m+1}})) \sim d(Q, \mathrm{fron}(U_{m+1}))$ for $Q \in N_j(S_{\lambda_{t+1}^{m+1}}, \mathrm{fron}(S_{\lambda_{t+1}^{m+1}}))$,

we obtain the first relations involving $I_{1k,j}(T_m^+)$ and $I_{1j}(U_m)$ in $(3.11)_4$ and $(3.11)_5$ respectively. Concerning the second relations in $(3.11)_{4,5}$ note the following facts.

(a_2) $d(Q, \mathrm{fron}(T_m^+)) \sim d(Q, \overline{S}_{\lambda^{m+1}}) \sim d(Q, \mathrm{fron}(T_m^+(S_{\lambda_1}, \ldots, \lambda_t)))$ for $Q \in N_j(S_{\lambda^{m+1}})$, λキル

(a'_2) $d(Q, \mathrm{fron}(U_m)) \sim d(Q, \overline{S}_{\lambda_t^{m+1}}) \sim d(Q, \mathrm{fron}(T_m^+(S_{\lambda_1}, \ldots, \lambda_t)))$

for $Q \in N_j(T_m^+(S_{\lambda_1}, \ldots, \lambda_t, \lambda_{t+1}^{m+1}))$.

Now the inclusions for $I_{2j}(T_m^+)$, $I_{2j}(T_m^+(S_{\lambda_1}, \ldots, \lambda_t, \lambda_{t+1}^{m+1}))$ in

$(3.11)_4$ and $(3.11)_5$ follow from $(a_2),(a_2')$ and from the obvious relations: $\mathrm{fron}(S_{\lambda_{t+1}^{m+1}}) \subset \overline{S}_{\lambda_{t+1}^{m+1}}$, $\mathrm{fron}(S_{\lambda^{m+1}}) \subset \overline{S}_{\lambda^{m+1}}$.

For homomorphisms $i_j(S),\dots,I_{2j}(T_m^+(S_{\lambda_1},\dots,\lambda_t))$, define a set of direst limits of homomorphisms as follows.

$$
(3.12)\begin{cases}
\{\hat{i}(S) = \varinjlim i_j(S)\},\{\hat{i}(T_1,T_2) = \varinjlim i_j(T_1,T_2)\}, \\
\hat{i}(T_1^+(S_{\lambda_1},\dots,\lambda_t)) = \varinjlim i_j(T_1^+(S_{\lambda_1},\dots,\lambda_t)), \\
\hat{I}_k(T_m^+) = \varinjlim I_{kj}(T_m^+), \quad \hat{I}_k(U_m) = \varinjlim I_{kj}(U_m); \\
U_m = T_m^+(S_{\lambda_1},\dots,\lambda_t), \quad k = 1,2.
\end{cases}
$$

In (3.12) $S,\dots\dots,U_m$ are as in (3.11). We let $K(\hat{\Omega}_{p.g.}(S,N))$ be the collection of homomorphisms $\{\hat{i}(S); S\in S\}$, $\{\hat{i}(T_1,T_2);\ T_1 \vee T_2 \in S_C\}$, $\{\hat{i}(T_1^+(S_{\lambda_1},\dots,\lambda_t))$; for any $T_1^+(S_{\lambda_1},\dots,\lambda_t)\}$, $\{\hat{I}_{kj}(T_m^+)$; $T\in S_C\}$, $\{I_{kj}(U_m)$; $U_m \in S_{OC}\}$, $k=1,2$. Then, for homomorphisms $\hat{i}(S)$, $\hat{i}(T_1,T_2)$, $\hat{i}(T_1^+(S_{\lambda_1},\dots,\lambda_t))$ in $K(\hat{\Omega}_{p.g.}(S,N))$, exact sequences for $\hat{C}'_{p.g.}(S)$, ..., $\hat{E}_{p.g.}(S_{\lambda_1},\dots,\lambda_t)$ corresponding to the exact sequences $(3.11)_{1,2,3}$ are valid. Therefore, if we prove the following facts (3.12)'s, then the collection $\hat{\Omega}_{p.g.}$ has a structure of a cochain complex attached to S with $K(\hat{\Omega}_{p.g.})$ in an obvious manner.

$(3.12)_1$ $\mathrm{Ker}\,\hat{I}_2(T_m^+) = \mathrm{image}\,\hat{I}_1(T_m^+)$; $T\in S_C$, and that $I_2(T_m^+)$ is surjective, where $T\in S_C$.

$(3.12)_2$ $\mathrm{Ker}(U_m$

$(3.12)_2$ Ker $\hat{I}_2(U_m)$ = image $\hat{I}_1(U_m)$, and that $\hat{I}_2(U_m)$ is surjective ,where $U_m \in S_{OC}$.

Let (U,V) be as in the beginning of § 3.3, and $\mathcal{S}$ a prestratification of V . Let $T \in S_C$,and let $S \in \mathcal{T}$. For a point $P \in S$, R^n_P and $|T|_P$ stand for the germs of the ambient space R^n and $|T|$ at P. We say that $\mathcal{T}$ is <u>complete</u> along S if the following is satisfied.

(☆) If at a point $P \in S$ $R^n_P \subset |T|_P$, then for any $P' \in S$ $R^n_P \subset |T|_P$. Moreover, if $S \subset \mathrm{int}(\mathcal{T})$, then $N_\delta(S, \mathrm{fron}(S)) \cap \mathcal{V} \subset |T|$ holds with a suitable (δ).

Let (U,V) be as above and S a P.G.-adequate prestratif--ication of V. Moreover, let N be a direct system of C^∞-thickenings of $\mathcal{S}$. Now we show the following lemma.

Lemma 3.1. <u>Let the triple(U,V,S) be just as above. Assume that ,for each pair $(S,T) \in \mathcal{S} \times S_C$ satisfying the relation $S \in \mathcal{T}$, the condition (☆) is valid. Then $(3.12)_{1,2}$ are valid , and the collection $\hat{\Omega}_{p.g.}(S,N)$ is a cochain complex attached to S with $K(\hat{\Omega}_{p.g.}(S,N))$.</u>

(by λ^{m+1})

of $(3.12)_1$ below. Then the proof of $(3.12)_2$ is applicable to that of $(3.12)_1$ word by word. Let F', G', H' be the families of analytic functions associated with S. Take an element $U_m = T_m^+(S_{\lambda_1}, \dots, \lambda_t) \in S_{OC}$. In the arguements below, for an element $\hat{\varphi}_{p.g.} \in \hat{\Omega}_{p.g.}(T_m^+)$, $\hat{\Omega}_{p.g.}(\langle(S_{\lambda_1}, \dots, \lambda_t)\rangle), \dots, \varphi_{p.g.j}$ stands for an element in $\Omega_{p.g.j}(\langle(T_m)\rangle), \dots, \Omega_{p.g.j}(\langle(S_{\lambda_1}, \dots, \lambda_t)\rangle)$ such that $\rho_{\infty j}(\varphi_{p.g.j}) = \hat{\varphi}_{p.g}$.

(I) First we show that $\ker \hat{I}_2(U_m) = \text{image} \hat{I}_1(U_m)$. Take an element $\hat{\varphi}_{p.g.}$ in $\ker(\hat{I}_2(U_m))$. Regarding $\hat{\varphi}_{p.g.}$ as an element in $\hat{\Omega}(U_m)$, we can write a representative $\varphi_{p.g.j}$ of $\hat{\varphi}_{p.g}$ in the following form.

$$(3.13) \qquad \varphi_{p.g.j} = I_{1j}(\varphi_j), \text{ where } \varphi_j \in \Omega(N_j(U_{m+1})) .$$

For our purpose, it is, then, sufficient to show that $\varphi_j \in \Omega_{p.g.j}(\langle(U_{m+1})\rangle)$, where $U_{m+1} = T_{m+1}^+(S_{\lambda_1}, \dots, \lambda_t)$. From the relation $N_j(U_{m+1}) = N_j(U_m) \cup \left\{ \bigcup_{\lambda_t}^{m+1} N_j(S_{\lambda_1}, \dots, S_{\lambda_{t+1}^{m+1}}) \right\}$ and (3.13) we know that

$$(3.14) \quad \varphi_j = \varphi_{p.g.j} \text{ in } N_j(U_m) \text{ and in } N_j(S_{\lambda_1}, \dots, \lambda_{t+1}^{m+1})$$

for each λ_{t+1}^{m+1}.

On the otherhand $d(Q, \text{fron}(U_m)) \sim d(Q, \overline{S}_{\lambda_{t+1}^{m+1}})$ in $N_j(S_{\lambda_{t+1}^{m+1}})$, and $d(Q, \overline{S}_{\lambda_{t+1}^{m+1}}) \succsim d(Q, \text{fron}(S_{\lambda_{t+1}^{m+1}}))$ outside $N_j(S_{\lambda_{t+1}^{m+1}})$. Therefore, we know that

$(3.14)'$ $d(Q,\mathrm{fron}(U_m)) \sim d(Q,\mathrm{fron}(U_{m+1}))$ in $N_j(U_m)$ $- \bigcup_{\lambda_{t+1}^{m+1}} N_j(S_{\lambda_{t+1}^{m+1}})$.

Moreover, $d(Q,\mathrm{fron}(S_{\lambda_{t+1}^{m+1}})) \sim d(Q,\mathrm{fron}(\breve{U}_{m+1}))$ in $N_j(S_{\lambda_{t+1}^{m+1}})$. It is clear that (3.13),(3.14),$(3.14)'$ and the distance comparison just above suffices to show that $\varphi_{p.g.j} \in I_1(\Omega_{p.g.j}(U_{m+1}))$.

(II) For $U_m = T_m^+(S_{\lambda_1},\ldots,\lambda_t)$, let U_m'' be the subset of U_m composed of those elements S's such that $S \not\subset \mathrm{int}|U_m|$. Of course U_m contains strata with dimension at most n-1. If U_m does not contain a stratum of dimension n, then $U_m = U_m'''$. Note that $U_m \subset S_C$ implies that $U_m'' \subset S_C$. We prove the surjectivity of the map $\widehat{I}_2(U_m)$ in the following two steps:

$(II)_1$ For an element $\widehat{\varphi}_{p.g.} \in \bigoplus_{\lambda_{t+1}^{m+1}} \widehat{\Omega}_{p.g.}(U_m(S_{\lambda_{t+1}^{m+1}}))$, take a representative $\varphi_{p.g.j}$ of $\widehat{\varphi}_{p.g.}$. For $\varphi_{p.g.j}$, define an element $\varphi'_{p.g.j}$ in $\bigoplus_{\lambda^{m+1}} \Omega_{p.g.j}'(S_{\lambda},\ldots,\lambda^{m+1})$ as follows:

$(3.15)_1$ For λ_{t+1}^{m+1} such that $U_m'''(S_{\lambda_{t+1}^{m+1}}) = \phi$, $\varphi'_{p.g.j} =$

$$\varphi_{p.g.j} \cdot \text{in } N_{j'}(S_{\lambda_1},\ldots,\lambda_{t+1}^{m+1}) .$$

(3.15) For λ_{t+1}^{m+1} such that $U(S_{\lambda_{t+1}^{m+1}}) \neq \phi$, $\varphi_{p.g.j} =$

$\varphi_{p.g.j}$ in int $[U''_m(S_{\lambda^{m+1}_{t+1}})] \cap N_{j'}(S_{\lambda_1},\ldots,{}_{\lambda^{m+1}_{t+1}})$, and $\varphi'_{p.g.j} =$
$\chi_a(f',h' : S_{\lambda^{m+1}_{t+1}}, U''_m(S_{\lambda^{m+1}_{t+1}}))\cdot\varphi_{p.g.j}$. in $N_{j'}(S_{\lambda_1},\ldots,\lambda^{m+1}_{t+1})$ —
int $(U_m(S_{\lambda^{m+1}_{t+1}}))$.

In the above, $j' > j$, and (a) is a suitable couple (cf. § 3.

2.) . By a suitable choice of (a) and by the condition (✫), we can assume that $\varphi'_{p.g.j} = \varphi_{p.g.j}$ in $\bigcup_{\lambda^{m+1}_{t+1}} N_{j'}(U_m(S_{\lambda^{m+1}_{t+1}}))$.
Moreover, we can assume that this form $\varphi'_{p.g.j}$ is C^∞-differentiable in $\bigcup_{\lambda^{m+1}_{t+1}}\{N_j(U_m(S_{\lambda^{m+1}_{t+1}})) - \bar{S}_{\lambda^{m+1}_{t+1}}\}$. Then $\varphi'_{p.g.j}$ is of polynomial growth with respect to $\bigcup_{\lambda^{m+1}_{t+1}} \bigcup \bar{S}_{\lambda^{m+1}_{t+1}}$.

(II)$_2$ Next we start with the form $\varphi'_{p.g.j}$ constructed in (II)$_1$. In this case, for a suitable couple (b), let $\varphi''_{p.g.j}$ be a form defined by

$$\varphi''_{p.g.j} = \sum_{\lambda^{m+1}_{t+1}} \chi_b(f',g' : S_{\lambda^{m+1}_{t+1}}) \cdot \varphi'_{p.g.j} .$$

Then for a suitable couple (b), we can assume that $\varphi''_{p.g.j}$
$= \varphi''_{p.g.j}$ in $N = \bigcup_{\lambda^{m+1}_{t+1}}\{N_{j''}(S_{\lambda^{m+1}_{t+1}}) - \bar{S}_{\lambda^{m+1}_{t+1}}\}$. Moreover,

$\varphi''_{p.g.j}$ is a C^∞-form in $N_{j''} \cup (U - \bigcup_{\lambda_{t+1}^{m+1}} N_{j'}(S_{\lambda_{t+1}^{m+1}}))$ by letting $\varphi''_{p.g.j} = 0$ outside $\bigcup_{\lambda_{t+1}^{m+1}} N_{j'}(S_{\lambda_{t+1}^{m+1}})$.

Moreover, define $\tilde{\varphi}''_{p.g.j}$ by the following equation.

$$(3.16) \qquad \tilde{\varphi}''_{p.g.j} = (1 - \chi_b(f', g' : S_{\lambda_{t+1}^{m+1}})) \cdot \varphi'_{p.g.j}$$

in each $N_{j'}(S_{\lambda_1}, \dots, \lambda_{t+1}^{m+1})$.

This function $\tilde{\varphi}''_{p.g.j}$ is defined in $\bigcup_{\lambda_{t+1}^{m+1}} N_{j''}(S_{\lambda_1}, \dots, \lambda_{t+1}^{m+1})$ and is of polynomial growth with respect to $\bigcup_{\lambda_{t+1}^{m+1}} \mathrm{fron}(S_{\lambda_{t+1}^{m+1}})$. Obviously $\varphi''_{p.g.j} + \tilde{\varphi}''_{p.g.j} = \varphi'_{p.g.j}$. Let $\hat{\varphi}^1_{p.g.} = \rho_{\infty j''}(\varphi_{p.g.j})$ and $\hat{\varphi}^2_{p.g.} = \rho_{\infty j''}(\tilde{\varphi}''_{p.g.j})$. Then $\varphi^1_{p.g.} \in \hat{\Omega}_{p.g.}(U_m)$ in view of the fact stated in the last of $(II)_1$. Also $\hat{\varphi}^2_{p.g.} \in \bigoplus_{\lambda_{t+1}^{m+1}} \hat{\Omega}_{p.g.}(S_{\lambda_1}, \dots, \lambda_{t+1}^{m+1})$ in view of the fact that $\varphi_{p.g.}$ is of polynomial growth with respect to $\bigcup \mathrm{fron}(S_{\lambda_{t+1}^{m+1}})$. Clearly $\hat{I}_2(U_m)(\hat{\varphi}^1_{p.g.} + \hat{\varphi}^2_{p.g.}) = \hat{\varphi}_{p.g.}$. This finishes our proof of Lemma 3.1.

We call the C.C.S. $\hat{\Omega}_{p.g.}(S,N)$ with $K(\hat{\Omega}_{p.g.}(S,N))$ in Lemma 3.1 will be called the (P.G.-C^∞-de Rham cochain complex attached to (S,N).

3.4. <u>Relations between $\widehat{\mathfrak{L}}(S,N)$ and $\widehat{\mathfrak{L}}_{p.g.}(S,N)$</u>. Let (U, V) and (S,N) be as in Lemma 3.1. Then we have two C.C.S.'s $\widehat{\mathfrak{L}}(S,N)$ and $\widehat{\mathfrak{L}}_{p.g.}(S,N)$. Let U denote an element in S_C, S_0 or S_{0C}. Note that, for each $j \in Z^+$ and U, there exists a canonical injection $i_{p.g.j}: \mathfrak{L}_{p.g.j}(U) \hookrightarrow \mathfrak{L}_j(U) = \mathfrak{L}(N_j(U))$. Then we can form direct limits $\hat{i}_{p.g.}(\hat{U}) : \widehat{\mathfrak{L}}_{p.g.}(U) = \widehat{\mathfrak{L}}_{p.g.}(S,N:U) \longrightarrow \widehat{\mathfrak{L}}_{\gamma}(U) = \widehat{\mathfrak{L}}(S,N:U)$ by $\hat{i}_{p.g.}(U) = \varinjlim i_{p.g.j}(U)$. Here $U \in S_C$, S_0 or S_{0C}. According to $U \in S_C$, S_0 or S_{0C}, we write $i_{p.g.}(U)$ as $\hat{\alpha}_{p.g.}(U)$, $\hat{\beta}_{p.g.}(U)$ and $\hat{\beta}'_{p.g.}(U)$. Moreover, $\hat{\alpha}^*_{p.g.}(S,N)$, $\hat{\beta}^*_{p.g.}(S,N)$ and $\hat{\beta}'^*_{p.g.}(S,N)$ denote the collections $\{\hat{\alpha}^*_{p.g.}(U); U \in S_C\}$, $\{\hat{\beta}^*_{p.g.}(U); U \in S_0\}$ and $\{\hat{\beta}'^*_{p.g.}(U); U \in S_{0C}\}$ respectively. It is obvious that $\hat{\alpha}^*(S,N)$, $\hat{\beta}^*(S,N)$ $\hat{\beta}'^*(S,N)$ commute with $K(\widehat{\mathfrak{L}}(S,N))$ and $K(\widehat{\mathfrak{L}}_{p.g.}(S,N))$.

Corollary to Lemma 3.1. <u>Let (U,V) and (S,N) be as above. Assume that the homorphism</u>

(3.16) $$\hat{\beta}^*(U): H^*(\widehat{\mathfrak{L}}_{p.g.}(U)) \longrightarrow H^*(\widehat{\mathfrak{L}}(U)) \ ; \ U \in S_0$$

<u>is isomorphism. Then, for each $T \in S_C$ we have the following</u>

(3.17) $$H^*(\widehat{\mathfrak{L}}_{p.g.}(T)) \xrightarrow[\sim]{\hat{\alpha}^*(T)} H^*(\widehat{\mathfrak{L}}(T)) \xrightarrow{\sim} H^*(|T|;R)$$

Proof. The first isomorphism follows from (3.16) and Proposition 1.1. The second isomorphism follows from the difinition of direct systems of C^∞- thickengs (§ 2) and the obvious fact $N_{jT}(T) = |T|$ for each j.

§7. Proof of Theorem 6.1_d; d=1,...

The purpose of this section is to prove Theorem 6.1_d (d=1,..) inductively on d. Our arguments of this section are divided into the following two steps.

(I) To prove Theorem 6.1_1.

(II) To show that the validity of Theorem $6.1_{d'}$, $d'=1,\ldots,d-1$, implies the validity of Theorem 6.1_d ($d\geq 2$).

This section consists of two parts: §7.1 and §7.2. First we show the induction step (II) in §7.1. The proof of Theorem 6.1_1 is easy and will be given in §7.2.

§7.1. Discussion of the induction step: Theorem $6.1_{d'}$ ($d'=1,\ldots,d-1$) $\longrightarrow$ Theorem 6.1_d. ($d\geq 2$).

§7.1 consists of subsections: §7.1.1~§7.1.2, and will be, roughly, divided into the following three patrs.

(A) §7.1.1.~§7.1.2. This part is preparatory for the later parts (B),(C): In §7.1.1 we do a reduction of the induction step (II). In §7.1.2 we fix certain notations and data used in (B),(C).

(B) §7.1.3.~§7.1.10. Here we start with a given and fixed adequate series R at a point $P^n \in R^n(x)$. We then construct collections from R, denoted by $\mathbb{R}(r)$, $\mathbb{F}(r)$, of (i) Ser-

-ies of euclidean spaces, bounded domains, varieties and prestratifications and (ii) series of sets of functions. Here r denotes a series of positive numbers parametrizing collections R(r), $\boxed{F}$(r). Moreover, we construct collections R(r,r'), $\boxed{F}$(r,r'), parametrized by series r,r' of positive num--bers, from R(r)'s and $\boxed{F}$(r)'s by a simple divice.

(C) §7.1.10~§7.1.? . The main purpose of this part is to show that collections R(r,r'), $\boxed{F}$(r,r') are, under mild con--ditions on series r,r', normalized series attached to the adequate series R.

Further explanations of our arguments will be, when we feel necessary, inserted in parts (A),(B) and (C).

In §7.2.1. ~§7.2.10 ,<u>we fix an integer d($\geq$2) once and for all and assume the validity of Theorem $6.1_{d'}$ for d'=1,...,d-1</u>.

A. Preliminaries.

7.1.1. Here we make a simple remark about the induct-

-ion step (II) as follows. We divide adequate series R s of dimension d into the following two types.

(i) Adequate series R's satisfying the equality: dim R = rank R.

(ii) Adequate series R's satisfying the inequality: dim R $<$ rank R.

Let us assume that the assertions in Theorem 6.1_d are true for adequate series R's of type (i). We show, then, the validity of the assertions in Theorem 6.1_d also for adequate series R's of type (ii). To see this take an adequate series $\boxed{R} = \{Ⓡ, Ⓥ, Ⓥ', Ⓦ\}$ at a point P^n in a euclidean space $R^n(x)$, such that rank $\boxed{R} >$ dim $\boxed{R}$. We write Ⓡ, Ⓥ, Ⓥ' and Ⓦ explicitly as fo--llows : $Ⓡ = \{R^{k+1}(y_1', .., y_{k+1}'), .., R^n(y_1', .., y_n')\}$, $Ⓥ = \{V^j\}$, $Ⓥ' = \{V'^j\}$ and $Ⓦ = \{W^j\}$, j=k+1,..,n. Then we can define , in view of Proposition 5.1 , an adequate series $\boxed{\tilde{R}} = \{\tilde{Ⓡ}, \tilde{Ⓥ}, \tilde{Ⓥ}', \tilde{Ⓦ}\}$ as foll-ows.

(a) The series R is of the form: $\tilde{Ⓡ} = \{R^{d+1}(y_1'', .., y_{d+1}''), .., R^n(y_1'', .., y_n'')\}$, where $(y_1'', .., y_{k+1}'')$ is a linear transfoᵣm-ation of $(y_1', .., y_{k+1}')$ and $(y_{k+2}'', .., y_n'') = (y_{k+2}', .., y_n')$. (In the above d=dim R.)

(b) $\tilde{V}^j = V^j$, $\tilde{V}'^j = V'^j$ and $\tilde{W}^j = W^j$, j=k+1,..,n. Moreover, $\tilde{V}^j = \mathbb{L}^{an}_{jk+1}(y'')(V^{k+1})$, $V'^j = \mathbb{L}^{an}_{jk+1}(y'')(V'^{k+1})$, and $\tilde{W}^j = \phi$, j=d+1, ..,k.

The adequate series $\boxed{R}$ satisfies the equality: dim $\boxed{\tilde{R}}$ = d rank $\widetilde{\mathbb{R}}$. Therefore, we can choose a normalized series $(\widetilde{\mathbb{R}},\widetilde{\mathbb{F}})$ attached properly to $\widetilde{\mathbb{R}}$. Moreover, we can assume that the con ドイツ語 -ditions $(6.8)_{1,2}$ are valid for $\widetilde{\mathbb{R}}$. It is clear that, the norm- -alized series $(\widetilde{\mathbb{R}},\widetilde{\mathbb{F}})$ is attached properly to the original se- -ries $\widetilde{\mathbb{R}}$. Also it is clear that the conditions $(6.8)_{1,2}$ are valid for the series R. This shows our desired fact.

By §7.1.1. we restrict our attension to those adequate series R's satisfying the equality dim $\boxed{R}$ = rank $\boxed{R}$. Let P^n be a point in a euclidean space $R^n(x)$ and $\boxed{R}$ an adequate se- -ries at P^n, such that dim $\boxed{R}$ = rank $\boxed{R}$. We write $\boxed{R}$ explicit- -ly as follows : $\boxed{R} = \{Ⓡ, Ⓥ, Ⓥ', Ⓦ\}$, where $Ⓡ = \{R^{k+1}(y_1', \ldots, y_{k+1}'), \ldots, R^n(y_1', \ldots, y_n')\}$, $Ⓥ = \{V^j\}$, $Ⓥ' = \{V'^j\}$ and $Ⓦ = \{W^j\}$, $j=k+1, \ldots, n$. In §7.1.2 ~ §7.1.2, <u>we fix the above data</u> $R^n(x)$, P^n <u>, and</u> $\boxed{R} = \{Ⓡ, Ⓥ, Ⓥ', Ⓦ\}$ <u>once and for all. Our arguments in</u> §7.1.2~§7,1.10 <u>will be done fot these fixed data</u> $R^n(x)$, P^n <u>and</u> $\boxed{R}$.

~~7.1.2.~~

§ 7.1.2. Here we shall fix for the adequate series $\boxed{R}$ cer- -tain notations and data used in the later arguments.

(i) For $j=k+1, \ldots, n$, let $Ⓥ^j$ and $Ⓥ'^j$ denote the sets of all the irreducible components of V^j and V'^j. Moreover, we denote by $Ⓥ^j(d)$ and $Ⓥ'^j(d)$ the sets of all the irreducible components of dimension d of V^j and V'^j. Furthermore, we de- -note by $Ⓥ^j(d-1)$ and $Ⓥ'^j(d-1)$ respectively $Ⓥ^j - Ⓥ^j(d)$ and $Ⓥ'^j - Ⓥ'^j(d)$. We write the unions $Ⓥ^j \vee Ⓥ'^j$, $Ⓥ^j(d) \vee Ⓥ'^j(d)$ and $Ⓥ^j(d-1) \vee Ⓥ'^j(d-1)$ respectively as $\widetilde{Ⓥ}^j$, $\widetilde{Ⓥ}(d)$ and $\widetilde{Ⓥ}^j(d-1)$.

(ii) We denote by O^j the ring of germs of holomorphoc functions at $P^j = \pi_{jn}(y')(P^n)$, $j=k+1,..,n$. For a germ X^j of a variety at P^j, I_{X^j} will denote the ideal of X^j O^j.

(iii) For each germ $X^j_\lambda \in \tilde{V}^j(d)$, define a proper subgerm $Y_0(X^j_\lambda)$ of X^j_λ by

$$Y_0(X^j_\lambda) = X^j_\lambda \cap \left\{ \bigcup_{\lambda} X^j_{\lambda} \cup \bigcup_{\mu} W^j_{\mu} \right\} \quad ,j=k+1,..,n.$$

In the above X^j_{λ} and W^j_{μ} exhaust respectively all the germs in $V^j - \{X^j_\lambda\}$ and $W - \{X^j_\lambda\}$.

(iv) For each $X^j_\lambda \in \tilde{V}^j(d)$, <u>fix</u> a finite basis $\tilde{f}'(X^j_\lambda)$ of the ideal I_{X^j} <u>once and for all</u>. Moreover, <u>fix</u> a proper subgerm $Y(X^j_\lambda, f'(X^j_\lambda))$ of X^j_λ such that $X^j_\lambda - Y(X^j_\lambda, f(X^j_\lambda))$ is $\{(y'^{k+1}, y'^{k+2,j}), f'(X^j_\lambda)\}$-smooth. (Cf. § 5.)

Symbols $\tilde{V}(d), V(d-1), \tilde{V}, ..$ as well as data $\tilde{f}'(X^j_\lambda)$'s, $Y_0(X^j_\lambda, \tilde{f}'(X^j_\lambda))$, ,... as above will be used in the sequel of § 7.1.

B. Construction of collections R(r),F(r).

Here we shall construct collections R(r),F(r), paramet-rized by series r of positive numbers, of series explained in the beginning of § 7.1. This part B contains subsections 7.1.3.~ 7.1. 9 ,and contexts of this part can be subdivided as follows.

$(B)_1$ Construction of an adequate series, denoted by $\tilde{R}(d-1) = \{\tilde{R}, \tilde{V}(d-1), \tilde{V}'(d-1), \tilde{W}(d-1)\}$. Here $\tilde{V}(d-1)$, $\tilde{V}'(d-1)$ and $\tilde{W}(d-1)$ contain only germs of dimension at most d-1. (7.1.3~ 7.1.9)
In the construction of $\tilde{R}$, we use Proposition 5.1.

B_2. Construction of collections $\widehat{R}(r)$ $\widehat{F}(r)$ based on data $\widehat{R}(d-1)$ and $\{\widehat{V}^j(d)\}_{j=k+1}^{n}$, $\{f'(X^j_\lambda),\ X^j_\lambda \in V^j(d)\}_{j=k+1}^{n}$, and construct--ion of collections R9r,r'), $\widehat{F}(r,r')$ from $\widehat{R}(r)$'s and $\widehat{F}(r)$'s. (§ 7.1.9 ~ 7.1.9.)

In B_1 arguments are purely local in the sense that argu-ments are done for only germs of varieties anf functions. We will concern in B_2 with bounded domains, varieties,...

7.1.B_1 Construction of an adeqaute series $\boxed{R}(d-1)$.

7.1.3. (1) Here we shall associate with each $X^{k+1}_\lambda \in \widetilde{V}^{k+1}(d)$ a proper subgerm $Y'(X^{k+1}_\lambda)$ of X^{k+1} satisfying the following conditions.

$(7.1)_1$ For each $X^j_\lambda \in \widetilde{V}^{j'}(d)$, the inclusion $Y'(X^j_\lambda) \supset Y_0(X^j_\lambda) \cup Y(X^j, f'(X^j_\lambda))$ holds, $j=k+1,..,n$. Here we write the intersection $X^j \cap \pi^{-1}_{k+1j}(X^{k+1}_\lambda)$ with $X^{k+1}_\lambda = \pi^{an}_{k+1j}(X^j_\lambda)$ as $Y'(X^j_\lambda)$.

$(7.1)_2$ For any pair $(X^{j'}, X^j) \in V^{j'}(d) \times V^j(d)$ such that $\pi^{an}_{jj'}(X^{j'}) = X^j$, we have the following relation.

$$\pi^{an}_{jj'}: X^{j'}_\lambda - Y'(X^{j'}_\lambda) \xrightarrow{\text{loc. biho.}} X^j_\lambda - Y'(X^j_\lambda)$$

We show a method to associate with each $X^{k+1}_\lambda \in \widehat{V}^{k+1}(d)$ a proper subgerm $Y'(X^{k+1}_\lambda)$ssatisfying $(7.1)_{1,2}$:

$(1)_1$ With each pair $(X^j_\lambda, X^{j'}_{\lambda'}) \in \widehat{V}^j(d) \times \widehat{V}^{j'}(d)$ such that $\pi^{an}_{jj'}(y')(X^{j'}_{\lambda'}) = X^j_\lambda$, $k+1 \leq j \leq j' \leq n$, we associate a proper subgerm $Y(X^j_\lambda, X^{j'}_{\lambda'})$ of X^j_λ so that the following are valid.

$(7.1)_1'$ $Y(X^j_\lambda, X^{j'}_{\lambda'}) \supset Y_0(X^j_\lambda) \cup Y(X^j_\lambda, f'(X^j_\lambda))$, and moreover, $Y(X^{j'}_{\lambda'}, X^j_\lambda) \supset Y_{0_1}(X^{j'}_{\lambda'}) \cup Y(X^{j'}_{\lambda'}, f'(X^{j'}_\lambda))$, where we write the inter-section $X^j \cap \pi_{jj'}'(Y(X^j_\lambda, f'(X^j_\lambda)))$.

$(7.1)_2'$ $\pi_{jj'}': X^{j'} - Y(X^{j'}_{\lambda'}, X^j_\lambda) \xrightarrow{\text{loc.biho.}} X^j_\lambda - Y(X^j_\lambda, X^{j'}_{\lambda'})$

$(i)_2$ Next we define, for each $X^j_\lambda \in V^j(d)$, a proper subgerm $Y''(X^j_\lambda)$ of X^j_λ by

$(7.\ldots)$ $Y''(X^j_\lambda) = \bigcup_{j'} \pi^{an}_{jj'}(X^{j'}_{\lambda'})$, where $X^{j'}$ exhaust all the germs in $V^{j'}(d)$ such that $\pi^{an}_{jj'}(X^{j'}) = X^j_\lambda$.

$(i)_3$ Now we define, for each $X^{k+1}_\lambda \in V^{k+1}(d)$, a proper subgerm $Y'(X^{k+1}_\lambda)$ of X^{k+1}_λ by

(b) $Y'(X^{k+1}_\lambda) = \bigcup_{j \geq k} \pi^{an}_{k+1j}(y')(X^j_\lambda)$, where X^j_λ exhaust all the germs in $V^j(d)$ such that $\pi^{an}_{k+1j}(X^j_\lambda) = X^{k+1}_\lambda$.

Then it is clear that the proper subgerm $Y'(X^{k+1}_\lambda)$ as above of $X^{k+1}_\lambda \in V^{k+1}(d)$ satisfies $(7.1)_{1,2}$.

We fix, for each $X^{k+1}_\lambda \in V^{k+1}(d)$, a proper subgerm $Y'(X^{k+1}_\lambda)$ of X^{k+1} satisfying $(7.1)_{1,2}$ in the later arguments.

(ii) Now we define a germ $\tilde{V}'^{k+1}(d-1)$ at P^{k+1} by

$$(7.2) \qquad \tilde{V}'^{k+1}(d-1) = \tilde{V}^{k+1}(d-1) \cup \{\bigcup_\lambda Y'(X^{k+1}_\lambda)\},\ X^{k+1}_\lambda \in V^{k+1}(d)$$

Clearly $\dim V'^{k+1}(d-1) \leq d-1$. Moreover, it is obvious that each germ Y^{k+1}_μ of $\tilde{V}^{k+1}(d-1)$ is an irreducible compon-ent of $\tilde{V}'^{k+1}(d-1)$.

7.1.4. Here we use Proposition 5.1 to proceed one more step: By Proposition 5.1 there exists a bijective linear map L^{k+1} of $R^{k+1}(y'_1,..,y'_{k+1})$: $L(y'_1,..,y'_{k+1}) = (y''_1,..,y''_{k+1})$ so that the following are valid.

$(7.3)_1$ For any $Z^{k+1} \subset V^{k+1}$, the map $\pi^{an}_{kk+1}: Z^{k+1} \to \pi^{an}_{kk+1}(Z^{k+1})$ is integral, where we abbriviate $\pi^{an}_{kk+1}(y'')$ as $\pi^{an''}_{kk+1}$.

$(7.3)_2$ For any pair $(Y^{k+1}_{\mu} \neq Y^{k+1}_{\mu'})$ of irreducible components of $V'^{k+1}(d-1)$, $\pi^{an}_{kk+1}(Y^{k+1}_{\mu}) \neq \pi^{an}_{kk+1}(Y^{k+1}_{\mu'})$.

$(7.3)_3$ For any pair $(Y^{k+1}_{\mu}, X^{k+1}_{\lambda}) \in V^{k+1}(d-1) \times V^{k+1}(d)$,

$\pi^{an}_{kk+1}(Y^{k+1}_{\mu}) \not\subset \pi^{an}_{kk+1}(R(X^{k+1}_{\lambda}))$, where we put $R(X^{k+1}_{\lambda}) = R(X^{k+1}_{\lambda}, f(X^{k+1}_{\lambda}))$ (cf. §6

We fix a bijective linear map L^{k+1} satisfying $(7.3)_{1,2}$, and the resulting cooydinates $(y''_1,..,y''_{k+1}) = L(y'_1,..,y'_{k+1})$ in the later arguments. For notational reasons we write the system of coordinates $(y''_1,..,y''_{k+1}, y'_{k+2},..,y'_n)$ as $(y''_1,..,y''_{k+1}, y''_{k+2},..,y''_n)$ and abbriviat projections $\pi_{jj'}(y'')$ as $\pi''_{jj'}$.

The coordinates (y'') and the projections $\pi''_{jj'}$ will be utillized in §7.1.5.~§7.1.6.

Now we define a germ $V^k(d-1)$ at $P^k = \pi''_{kk+1}(P^{k+1})$ by

(7.4)
$$V^k(d-1) = \pi^{an''}_{kk+1}(Y'^{k+1}(d-1)) \cup \bigcup_{\lambda} \pi^{an''}_{kk+1}(R(X^{k+1}_{\lambda}))$$
where $X^{k+1}_{\lambda} \subset V^{k+1}(d)$.

Our arguments from now on in (B) will be based on this $V^k(d-1)$.

7.2.5. (i) Now we define germs $\tilde{V}^j(k-1)$ at P^j $(j=k+1,\ldots,n)$ by

$$(7.5)_1 \qquad \tilde{V}^j(d-1) = \pi_{kj}^{\prime\prime -1}(V^k(d-1)) \cup \overset{\mu\nu}{V^j}.$$

Moreover, define germs $\tilde{\tilde{V}}^j(d-1)$, $\tilde{V}'^j(d-1)$ at P^j $(j=k,\ldots,n)$ by

$$(7.5)_2 \qquad \tilde{\tilde{V}}^j(d-1) = \tilde{V}^j(d-1) \quad, j=k+1,\ldots,n, \text{ and } \tilde{\tilde{V}}^k(d-1) = \pi_{kk+1}^{an''}(V^{k+1}(d-1)).$$

$(7.5)_3$ $\tilde{V}'^j(d-1) = \bigcup_\mu Y'^j_\mu$, where Y'^j_μ exhaust all the irre-ducible components of $V^j_{(d-1)}$ that are not contained in $\tilde{V}^j_{(d-1)}$ $(j=k,\ldots,n)$

(ii) To treat germs in question at P^k, we shall intro-duce the following symbols: We denote by R^k the germ $R^k(v''^k)$ at P^k. Moreover, we denote the set R^k consisting of the sin-gle germ R^k by $V^k(d)$. Furthermore, we write the germ Y^k_μ as $Y^k_\mu(R^k)$.

Now we summarize basic properties of $\tilde{V}^j(d-1), \tilde{V}^j(d),\ldots$

$(7.6)_1$ $\tilde{V}^j(d-1)$ and $\tilde{V}'^j(d-1)$ have no common irreducible components, $j=k,\ldots,n$.

$(7.6)_2$ For any irreducible component Y^j_μ of $\tilde{V}'^j(d-1)$, $\pi^{an}_{jj'}(Y^j_\mu)$ is an irreducible component of $\tilde{\tilde{V}}^j(d-1)$, $j=k,\ldots,n$.

$(7.6)_3$ $\widetilde{V}^{j'}(d-1) = \pi''^{-1}_{jj'}(\widetilde{V}^j(d-1)) \cap \widetilde{V}^{j'}$, $k \leq j \leq j' \leq n$.

$(7.6)_4$ For any pair $(X^j_\lambda, X^{j'}_{\lambda'}) \in \textcircled{V}^j(d) \times \textcircled{V}^{j'}(d)$ such that $\pi^{an}_{jj'}(X^{j'}_{\lambda'}) = X^j_\lambda$ $(k \leq j \leq j' \leq n)$,

$(7.6)_{4.1}$ $Y(X^{j'}_{\lambda'}) \supsetneq \pi^{-1}_{jj'}(Y(X^j_\lambda)) \cap X^{j'}_{\lambda'}$.

and

$(7.6)_{4.2}$ $\pi_{jj'}: X^{j'}_{\lambda'} - Y(X^{j'}_{\lambda'}) \xrightarrow{\text{loc.biho.}} X^j_\lambda - Y(X^j_\lambda)$.

$(7.6)_5$ For any $X^j_\lambda \in \textcircled{V}^j(d)$, $k+1 \leq j \leq n$,

$$Y(X^j_\lambda) \supset Y_0(X^j_\lambda) \cup Y(X^j_\lambda, f'(X^j_\lambda)).$$

①

Note that $(7.6)_{4,5}$ implies the following.

$(7.6)_6$ For each $X^j_\lambda \in \textcircled{V}^j(d)$, $j=k+1,..,n$,

$$\Delta(f'(X^j_\lambda);(y''^k, y''^j)) \cap (X^j_\lambda - Y(X^j_\lambda)) \neq \phi.$$

The above properties of germs $X^j, Y^j, \ldots$ follows easily from (7.1) and the definitions of $X^j, Y^j, \ldots$, and will be used arguments in B_2.

In the next subsection § 7.1.6 we shall define finite sets $W^j (j=k,..,n)$ of germs. Arguments in § 7.1.6 will be basi for the disscusion concerning the higher discriminant cond-tion in the part C.

7.1.6. (1) Recall that, for each $X^j_\lambda \in \textcircled{V}^j(d)$, the projec-tion $\pi^{an}_{kj}: X^j_\lambda \hookrightarrow R^k$ is integral $(j=k+1,..,n)$. Therefore we can attach to each $X^j_\lambda \in \textcircled{V}^j(d), j=k+1,..,n$, a set $f(X^j_\lambda) = f'_t(X^j_\lambda : y''^k)$

satisfying the following conditions.

(a) [illegible] $f(X^j_\lambda) \subset I_{X^j}$.

(b) The discriminant of $f_t(X^j_\lambda) \neq 0, t=1,..,j-d$.

We ~~fix~~, for each $X^j_\lambda \in V^j(d), j=k+1,..,n$, a set $f(X^j_\lambda)$ of Weierstrass polynomials at P^j (with respect to the coordinate (y'')), once and for all We note that, for each $X^j_\lambda \in V^j(d)$

(c) $\{V(f(X^j_\lambda)) - (f(X^j_\lambda),(y''^k, y''^j))\} \cap X^j_\lambda - Y(X^j_\lambda) \neq \phi, j=k+1,..$

(ii) Let $X^j_\lambda \in V^j(d)$, $j=k+1,..,n$. For each $m \in Z^{+j-d}$, define a germ $W'_m(f(X^j_\lambda))$ at P^j by

(7.7)' $W'_m(f(X^j_\lambda))$ is the locus of the gemrs of function $D^{m_t} f_t(X^j)$, $t=1,..,j-d$ and $m'_t=0,..,m_t-1$. Here $D^{m_t}f_t(X^j)$ is defined by

$$D^{m_t}f_t(X^j_\lambda) = \partial^{m_t} f_t(X^j) / \partial y''^{m_t}_{k+t} .$$

Moreover, define a subgerm $W_m(f(X^j_\lambda))$ by

(7.7) $W_m(f(X^j_\lambda)) = W'_m(f(X^j_\lambda)) \cap Y(X^j_\lambda)$.

Furthermore, define a finite set $W(f(X^j_\lambda))$ of sub germs of $Y(X^j_\lambda)$ by the following requirement.

(7.8) A germ W^j of a variety at P^j is in $W(f(X^j_\lambda))$ if and only if, for a suitable $m \in Z^{+j-d}$, W^j is an irreducible component of $W_m(f(X^j_\lambda))$.

Let W^j $W(f(X^j_\lambda))$. Moreover, let $m_1=(m^1_{k+1},..,m^1_n)$ and $m_2 = (m^2_{k+1},..,m^2_n)$ be in Z^{+j-d}ssuch that W^j is an irreducible co component of both $W_{m_1}(f(X^j_\lambda))$ and $W(f(X^j_\lambda))$. Put $m_0=(m^0_{1+1},..,m$ where, $m^0_s = \max(m^1_t, m^2_t)(s=k+1,..,j)$. Then W^j is an irreducible component of $W_{m_0}(f(X^j_\lambda))$. This mean the existence of the element $m=m(W^j)$) characterized by the following properties.

$(7.8)_1$ W^j is an irreducible component of $W_m(f(X^j_\lambda))$,

$(7.8)_2$ The element m is maximal in Z^{+j-d} satisfying (7. in the following sense: If $W^j_1 \in W_{m_0}(f(X^j_\lambda))$, then $m' < m$.

We call this element $m=m(W^j)$ the <u>exponent</u> of W^j with respect to $f(X^j_\lambda)$.

(iii) We define a finite set $W^j(d-1)$, $j=k+1,..,n$ by

$$W^j(d-1) = \bigcup \{W(f(X^j_\lambda))\}, \text{ where } X^j_\lambda \in V^j(d).$$

Moreover, define a finite set $\widetilde{W}^j(d-1)$ of germs at P^j, $j=k,..,n$, by

$$\widetilde{W}^j(d-1) = W^j \cup W(d-1) \cup V^j(d) \bigcup Y(X^j_\lambda),\ X^j_\lambda \in V^j(d),\ k+1 \leq j,$$
$$\widetilde{W}^k(d-1) = \phi.$$

In the above we denote by $Y(X^j_\lambda)$ the set of all the irreducible components of the germ $Y(X^j_\lambda)$.

7.1.7. Now we summarize arguments in 7.1.3.~ 7.1.6.

(i) Define series $\widetilde{V}(d-1)$, $\widetilde{V}'(d-1)$ of germs by

$(7.9)_1$ $\widetilde{V}(d-1) = \{\widetilde{V}^j(d-1)\}^n_{j=k}$, $\widetilde{V}'(d-1) = \{\widetilde{V}'^j(d-1)\}^n_{j=k}$.

Moreover, define a series W of finite sets of germs by

$(7.9)_2$ $\widetilde{W}(d-1) = \{\widetilde{W}^j(d-1)\}^n_{j=k}$.

Furthermore, define a series $R(d-1)$ of euclidean spaces by

$(7.9)_3$ $R(d-1) = \{R^j(y''^j)\}^n_{j=k}$.

Then the collection $\widetilde{R}(d-1) = \{R(d-1), \widetilde{V}(d-1), \widetilde{V}'(d-1), \widetilde{W}(d-1)\}$

is, in view of (7.6), an adequate series at $P^n \in R^n(x)$.

(ii) For each $j=k+1,..,n$, define collections $\mathcal{F}^j(d)$, $\mathcal{F}'^j(d)$ of sets of germs of functions by

(7.9)$_4$ $\mathcal{F}^j(d) = \{f(X^j_\lambda), X^j_\lambda \in \mathcal{V}^j(d)\}$, $\mathcal{F}'^j(d) = \{f'(X^j_\lambda), X^j_\lambda \in \mathcal{V}^j(d)\}$.

Moreover, define series $F(d), F'(d)$ by

(7.9)$_5$ $\mathcal{F}(D) = \{\mathcal{F}^j(d)\}^n_{j=k+1}$, $\mathcal{F}'(d) = \{\mathcal{F}'^j(d)\}^n_{j=k+1}$.

Data $\mathcal{R}(d-1)$, $\mathcal{F}(d)$ and $\mathcal{F}'(d)$ introduced above will basic in arguments in B_2.

The above 7.1.7 finishes arguments of B_1. Arguments in B_2 will bw based on the data $R(d-1), V(d), F(d)$ and $F'(d)$.

B_2 Construction of collections $R(r), F(r),..$

7.1.8. Now we apply Theorem $6.1_{d'}$ $(d'=1,..d-1)$ to the adequate series $\mathcal{R}(d-1)$ to obtain a normalized series $\mathcal{R}(d-1), \mathcal{F}(d-1)$ attached properly to $\mathcal{R}(d-1)$. We assume that t the normalized series $(\widetilde{\mathcal{R}}(d-1), \widetilde{\mathcal{F}}(d-1))$ is of monomial type and satisfies the differentiablity condition for $R(d-1)$. We write the normalized series $(\widetilde{\mathcal{R}}(d-1), \widetilde{\mathcal{F}}(d-1))$ explicitly as follows:
$\{\widetilde{\mathcal{R}}(d-1), \widetilde{\mathcal{F}}(d-1)\} = \widetilde{\mathcal{R}}(d-1), (\widetilde{\mathcal{U}}(d-1), \widetilde{\mathcal{V}}(d-1), \widetilde{\mathcal{S}}_0(d-1)), (\widetilde{\mathcal{U}}'(d-1), \widetilde{\mathcal{V}}'(d-1), \widetilde{\mathcal{S}}'_0(d-1))$, $(\widetilde{\mathcal{F}}(d-1), \widetilde{\mathcal{F}}'(d-1))$. We use the letters $(y_1,..,y_n)$ for the system of coordinates defining the series $\widetilde{\mathcal{R}}(d-1)$. Then

we know that

(7.10) $(y_{k+2},..,y_n) = (y'_{k+2},..,y'_n)$, and $(y_1,..,y_{k+1})$ is a linear transformation of $(y'_1,..,y'_{k+1})$.

Moreover, $\tilde{S}(d-1)$, $\tilde{S}'(d-1)$ will mean, respectively, the series of prestratifications of $\tilde{V}(d-1)$, $\tilde{V}'(d-1)$ induced from $\tilde{S}_\emptyset(d-1)$, $\tilde{S}'_0(d-1)$.

We will fix the normalized series $(\hat{R}(d-1), \hat{F}(d-1))$ as above once and for all. We also use the letters $(y_1,..,y_n)$ and symbols $S(d-1)$, $S'(d-1)$ as above in the sequel of § 7.1.

7.1.9. Now we construct collections $\tilde{R}(r)$, $\tilde{F}(r)$ in the following divices:

(1) Choose a series $M = \{M^j\}_{j=1}^n$ of positive monomials so that (1) for each $X^j_\lambda \in V^j(d)$, $f(X^j_\lambda)$ is $\{(y^j) M^j\}$- estimatsd, $(j=k+1,..,n)$ and (2) the normalized series $(\hat{R}(d-1), \hat{F}(d-1))$ is of type M. Moreover, we choose a series r'_0 of positive nu--mbers of type M so that the following are valid.

$(7.11)_1$ The series $U(P^n,y,r)$ is consistent with $(R(d-1),F(d-1))$.

$(7.11)_2$ Each germ $X^j_\lambda \in V^j(d)$, $j=k+1,..,n$, has a represen--tative X^j_λ of X^j_λ, $f(X^j_\lambda)$ of $f(X^j_\lambda)$ and $f'(X^j_\lambda)$ of $f'(X^j_\lambda)$ in

$\mathfrak{U}(P^n, y, x)$.

We fix the series M, r_0', chosen as above, in the sequel of §7.1. Moreover, we will fix, for each $X_\lambda^j \in \mathcal{V}^j(d)$ $(j=k+1,\ldots,n)$, representatives $X_\lambda^j, f(X_\lambda^j)$ and $f'(X_\lambda^j)$, respectively, of $X_\lambda^j, f(X_\lambda^j)$ and $f'(X_\lambda^j)$ in $U^j(P^n,y,r)$ once and for all. We write $f(X_\lambda^j), f'(X_\lambda^j)$ as $f(X_\lambda^j), f'(X_\lambda^j)$, $X_\lambda^j \in \mathcal{V}^j(d)$ $(j=k+1,\ldots,n)$. Further-more, we will write the collections $\{X_\lambda^j\}, \{f(X_\lambda^j)\}, \{f'(X_\lambda^j)\}$, $X_\lambda^j \in \widetilde{\mathcal{V}}^j(d)$ $(j=k+1,\ldots,n)$ as $\mathcal{V}^j(d), \widetilde{\mathcal{F}}^j(d)$ and $\widetilde{\mathcal{F}}'^j(d)$.

(ii) Let $r < r_0'$ be a series of positive numbers of type M. For each $j=1,\ldots,n$, we shall make the following convention (a) For a subset A^j of $R^j(y^j)$, we write the intersection $A^j \cap U^j(P^n,y,r)$ as $A^j(r)$. (b) Let g^j be a function defined in a domain containing $U^j(P^n,y,r)$. When we emphasize $U(P^n,y,r)$ we write the restriction of g^j to $U^j(P^n,y,r)$ as $g^j(r)$. (c) for collections $A = A^j$ of subsets in $R^j(y^j)$ and $G = g^j$ of functions defined in domains containing $U^j(P^n,y,r)$, we arite $A^j(r)$ and $g^j(r)$ as $A(r)$ and $G(r)$.

(ii)'' Let $X_\lambda^j \in \mathcal{V}^j(d)$, $j=k+1,\ldots,n$. We write $f(X_\lambda^j)(r)$ and $f'(X_\lambda^j)(r)$ as $f(X_\lambda^j(r))$ and $f'(X_\lambda^j(r))$. Moreover, for the germ $Y(X_\lambda^j)$ we write $Y(X_\lambda^j)(r)$ as $Y(X_\lambda^j(r))$. Here $Y(X_\lambda^j)$ denotes the representative of $Y(X_\lambda^j)$ in $(\widetilde{\mathcal{U}}^j, \mathcal{R}(d-1), \mathcal{F}(d-1))$. (Cf. § 6). Furthermore, we write $\widetilde{\mathcal{F}}^j(d)(r), \widetilde{\mathcal{F}}'(d)(r)$ as $\widetilde{\mathcal{F}}^j(d,r), \widetilde{\mathcal{F}}'^j(d,r)$, $j=k+1,\ldots,n$. Finally, for j-th components $\widetilde{\mathcal{U}}^j(d-1), \widetilde{\mathcal{Y}}^j(d-1)$, $\mathcal{S}_0^j(d-1), \widetilde{\mathcal{U}}'^j(d-1), \ldots$ and $\widetilde{\mathcal{F}}^j(d-1), \widetilde{\mathcal{F}}'^j(d-1)$ $(j=1,\ldots,n)$, we write $\widetilde{\mathcal{U}}^j(d-1)(r), \widetilde{\mathcal{Y}}^j(d-1)(r), \ldots, \widetilde{\mathcal{F}}^j(d)(r), \ldots$ as $\widetilde{\mathcal{U}}^j(d-1,r), \mathcal{V}^j(d-1,$

$\ldots, \widetilde{F}^j(d-1,r), \ldots$ Similar abbriviations of symbols as above will be done, when there does not occur a confusion.

(iii) Now we choose a series $r_0 < r_0'$ of type (M) so that the following are valid.

$(7.12)_1$ For each $X^j_\lambda \in \widetilde{V}^j(d), j=k+1,\ldots,n,$

$(7.12)_{1\cdot 1}$ $X^j_\lambda(r_0)$ is the locus of $f'(X^j_\lambda(r_0))$,

and

$(7.12)_{1.2}$ $f(X^j_\lambda(r_0))$ vanishes on $X^j_\lambda(r_0)$ and is $U^j(P^n,y,r_0)$-
-estimated.

Moreover,

$(7.12)_{1.3}$ $Y(X^j_\lambda(r_0))$ is a subvariety of $X^j_\lambda(r_0)$,

and

$(7.12)_{1.4}$ $X^j_\lambda(r_o) - Y(X^j_\lambda(r_0))$ is $\{(y^d,y^j),f'(X^j_\lambda)\}$-smooth.

$(7.12)_2$ For any pair $(X^j_\lambda, X^{j'}_\lambda) \in \widetilde{V}^j(d) \cup \widetilde{V}^{j'}(d)$ such that $\pi^{an}_{jj'}(X^{j'}_\lambda(d)) = X^j_\lambda(d)$, $k+1 \leq j \leq j' \leq n$,

$$\pi_{jj'}: X^{j'}_\lambda(r) - Y(X^{j'}_\lambda(r)) \xrightarrow{\text{loc.biho.}} X^j_\lambda - Y(X^j_\lambda(r)).$$

$(7.12)_2'$ For any $X^j_\lambda \in V^j(d)$, $j=k+1,\ldots,n$,

$$\pi_{jj'}: X^j_\lambda(r) - Y(X^j_\lambda(r)) \xrightarrow{\text{loc.biho.}} U^k(d-1,r).$$

(For the conditions (7.12), compare the conditions in §6.

The series r_0 of positive number as above eill be fixed once and for all.

(iv) Let $r<r_0$ be a series of positive numbers of type (M). We define a variety $V^j(r)$ by

(7.13) $\tilde{V}^j(r) = \tilde{V}^j(d,r) \cup \tilde{V}^j(d-1,r), j=k+1,\dots,n,$

$, j=1,\dots,k.$

Next, for each $X^j_\lambda(r) \in \tilde{V}^j(d,r)$ $(j=k+1,\dots,n)$, let $\tilde{S}(X^j_\lambda(r))$ denote the collection of all the connected components of $X^j_\lambda(r) - Y(X^j(r))$. We write the collections $\{S(X^j_\lambda(r)), X^j_\lambda \in V^j(d,r)\}$ as $\tilde{S}^j(d,r), j=k+1,\dots,n$. Moreover, let $\tilde{S}^j_c(r)$ denote the collection of all the connected components of $U^j(P^n,y,r) - \tilde{V}^j(r), j=k+1, \dots,\dots,$ Then we define collections $\tilde{S}^j(r), \tilde{S}^j_0(r)$ $(j=1,\dots,n)$ of analytic manifolds by

$(7.13)_2$ $\tilde{S}^j(r) = \tilde{S}^j(d-1,r), j=1\dots,k,$ and $\tilde{S}^j(r) = S^j(d-1,r) \cup \tilde{S}^j(d,r), j=k+1,\dots,n.$

$(7.13)'_2$ $\tilde{S}^j_0(r) = \tilde{S}^j_0(r), j=1,\dots,k,$ and $\tilde{S}^j_0(r) = \tilde{S}^j(r) \cup \tilde{S}^j_c(r),$ $j=k+1,\dots,n.$

Thirdly we define series $\tilde{F}^j(r), \tilde{F}'^j(r)$, $j=1,\dots,n$, of sets of functions by

$(7.13)_3$ $\tilde{F}^j(r) = \tilde{F}^j(r), j=1,\dots,k,$ and $\tilde{F}^j(r) = \tilde{F}^j(d-1,r) \cup \tilde{F}^j(d,r)$ $, j=k+1,\dots,n,$

$(7.13)'_3$ $\tilde{F}'^j(r) = \tilde{F}'^j(r), j=1,\dots,k,$ and $\tilde{F}'^j(r) = \tilde{F}'^j(d-1,r) \cup \tilde{F}'^j(d,r), j= k+1,\dots,n.$

Now we will write the series $\tilde{V}^j(r)$, $S^j(r)$, $S_0^j(r)$, $\tilde{F}^j(r)$ and $\tilde{F}'^j(r)$, $j=1,\ldots,n$, respectively as $\tilde{V}(r)$, $\tilde{S}(r)$, $\tilde{S}_0(r)$, $\tilde{F}(r)$ and $\tilde{F}'(r)$., and write the series $U(P^n, y, r)$ as $U(r)$. Moreover we will write the series $\{R^j(y^j)\}_{j=1}^n$ as R and the collection $\{U(r), V(r), S_0(r)\}$ as $Q(r)$.

Finally we define, for any series $r<r_0$ of type M, collections $R(r)$, $F(r)$ by

$$(7.14)_1 \quad R(r) = \{R, Q(r)\}, \quad \boxed{F}(r) = \{F(r), F'(r)\}.$$

Moreover, define, for any pair (r,r') of series of ~~posi-~~ positive numbers of type M such that $r<r'<r_0$, collections $R(r,r')$ and $\boxed{F}(r,r')$ by

$$(7.14)_2 \quad R(r,r') = \{R, Q(r), Q(r')\}, \quad \boxed{F}(r,r') = \boxed{F}(r').$$

It is these collections $(R(r), \boxed{F}(r))$, $(R(r,r'), \boxed{F}(r,r'))$ that we will concern with in the next part (C).