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Exceptional sets of algebraic varieties of hyperbolic type

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1. Let V be a projective n-manifold (i.e., non-singular projective algebraic variety of dimension n) of hyperbolic type. Assume that the m-canonical system $|mK(V)|$ has no base point for large m. We then have a morphism $f = \Phi_m : V \to W \subset \mathbb{P}^{N_m}$ where $N_m = P_m(V) - 1$. Choose m so large that f is birational and that W is normal. Let $\Sigma = \{w \in W ; \dim f^{-1}(w) > 0\}$, which is a Zariski closed subset of W. Σ is a union of irreducible components $\Sigma_1, \ldots, \Sigma_r$. Our purpose here is to prove that each Σ_j is an algebraic variety of elliptic type. We call Σ the total exceptional set and Σ_j an exceptional subvariety of V.

2. We prove this by induction on $\dim f(\Sigma_j)$. If $\dim f(\Sigma_j) > 0$, then consider a general hyperplane section W_1 of $W \subset \mathbb{P}^{N_m}$. $V_1 = f^{-1}W_1$ is an (n-1)-submanifold of V, which is linearly equivalent to $mK(V)$ as a divisor i.e., $V_1 \sim mK(V)$. By the symbol $\mathcal{O}(E)$ we indicate the sheaf of germs of holomorphic sections of the vector bundle or the divisor E. Considering the exact sequence

$$0 \to \mathcal{O}(eK(V)) \to \mathcal{O}((m+e)K(V)) \to \mathcal{O}((m+e)K(V)|V_1) \to 0$$

with the fact that $K(V_1) \sim (K(V)+V_1)\,|\,V_1 \sim (1+m)K(V)\,|\,V_1$, we obtain the exact sequence:

$$H^0(V, \mathcal{O}(me_1)K(V)) \to H^0(V_1, \mathcal{O}(me_1(V_1))) \to H^1(V, \mathcal{O}(e_1(m-1)K(V)))$$

where $e = m(e_1 - 1)$. By a generalization of Kodaira's vanishing theorem, $H^p(-mK(V)) = 0$ for any $m > 0$ and $p < n$. Hence, the Serre duality implies $H^1((m+1)K(V)) = 0$. Therefore, $Tr_{V_1}\,|e_1mK(V)|$ is complete, so $\Phi_{e_1mK(V)}\,|\,V_1 = \Phi_{e_1K(V)}$. Hence, $\Sigma_j \cap V_1$ is exceptional for $\Phi_{e_1K(V_1)}$. By induction hypothesis, it follows that $\kappa(\Sigma_j \cap V_1) = -\infty$, which induces $\kappa(\Sigma_j) = -\infty$. Thus we can assume that $f(\Sigma_j)$ is a point p and write $E = \Sigma_j$ which is a subvariety of codimension r.

3. If $r = 1$, by the generalized adjunction formula, we have

$$\kappa(E) \leq \kappa(\{K(V)+E\}\,|\,E,\ E) = \kappa(E\,|\,E,\ E).$$

A general hyperplane section W_1 does not contain p and $|W_1|_p$ has a member W'. Then

$$f^*W' = r_1E + G \sim f^*(W_1),$$

where each component of G differs from E and $r_1 > 0$. Hence

$$f^*(W_1)\,|\,E = r_1E\,|\,E + G\,|\,E_1 \sim f^*(W_1\,|\,p) = 0,$$

which implies $\kappa(E\,|\,E,\ E) = -\infty$. Thus $\kappa(E) = -\infty$.

4. In the case of $r > 1$, we first resolve the singularity of E following the method of Hironaka. Let $\mu_1 : V_1 \to V$ be a monoidal transformation with non-singular center $C \subset \operatorname{sing}(E)$, the singular locus of E. Let E_1 be the strict

transform and L_1 the exceptional divisor for μ_1. Then

$K(V_1) = \mu^* K(V) + (v_1 - 1)L_1$ and $K(V_1)\,|\,E_1 = \mu^*(K(V)\,|\,E) + (v_1 - 1)L_1\,|\,E$

where v_1 = codim. of C in V. If C is a divisor on E, then $v_1 - 1 = r$ and if not, $L_1\,|\,E_1$ is exceptional for $\mu_1\,|\,E_1 : E_1 \to E$. If E_1 is still singular, we have to perform another monoidal transformation $\mu_2 : V_2 \to V_1$. Repeating this process, we have a sequence of monoidal transformations $V_\ell \xrightarrow{\mu_\ell} V_{\ell-1} \longrightarrow \cdots \longrightarrow V_1 \xrightarrow{\mu_1} V$ and a non-singular E_ℓ, being the strict transform of $E_{\ell-1}$. Let $\varphi_j = \mu_j \circ \cdots \circ \mu_\ell : V_\ell \longrightarrow V_{j-1}$. Then we have $K(V_\ell) = \varphi_1^* K(V) + \sum (v_j - 1)\varphi_j^* L_{j-1}$. Moreover by an exact sequence

$$0 \longrightarrow \Omega^1_{E_\ell} \longrightarrow \Omega^1_{V_\ell}\,|\,E_\ell \longrightarrow \mathcal{O}(N^\vee_{V_\ell/E_\ell}) \longrightarrow 0$$

where $N^\vee_{V_\ell/E_\ell}$ is the dual bundle of the normal bundle, we have

$$K(E_\ell) = K(V_\ell)\,|\,E_\ell + [\det N_{V_\ell/E_\ell}].$$

In order to compute the divisor $[\det N_{V_\ell/E_\ell}]$ we blow V_ℓ up with center E_ℓ. Then we have a monoidal transformation ψ: $V^* \longrightarrow V_\ell$, whose exceptional divisor L^* is isomorphic to $\mathbb{P}(N_{N_\ell/E_\ell})$. $\mathcal{O}(L^*)\,|\,L^*$ is the dual of the fundamental sheaf $\mathcal{O}_{\mathbb{P}}(1)$ of $\mathbb{P} = \mathbb{P}(N_{V_\ell/E_\ell})$. Consider a general hyperplane section D^* of V^* and write $D = f\,\varphi_1(D^*)$, which is a prime divisor passing through p. Write $f^*D = D_0 + \cdots$, D_0 being the proper transform of D. Then since codim $E = r > 1$,

D_0 contains E. Letting ε_1 be the order of D_0 at the generic point of C, we have $\mu_1^* D_0 = D_1 + \varepsilon_1 L_1$, where D_1 is the strict transform of D. Considering in the same way as a above, we have

$$\varphi_1^* D_0 = D_\ell + \varepsilon_1 \varphi_1^* L_1 + \varepsilon_2 \varphi_2^* L_2 + \cdots + \varepsilon_\ell L_\ell,$$

$$\psi_1^* D = \psi^* \varphi_1^* D_0 = D^* + \varepsilon_1 \psi_1^* L_1 + \varepsilon_2 \psi_2^* L_2 + \cdots + \varepsilon_\ell \varphi^* L_\ell + \varepsilon L^*$$

where $\psi_j = \varphi_j \circ \psi$, and ε is the order of D at the general point of E_ℓ. Hence $\varepsilon_j \geqq \varepsilon$. Since $\mathcal{O}(\varepsilon L^*) \mid L^* \cong \mathcal{O}_{\mathbb{P}}(-\varepsilon)$, we have $\pi_*(\mathcal{O}(-\varepsilon L^*) \mid L^*) \cong S^\varepsilon(\mathcal{O} N^\vee_{V_\ell/E_\ell})$, where $\pi : L^* = \mathbb{P}(N_{V_\ell/E_\ell}) \to E_\ell$ is the natural projection. Moreover,

$$\psi_1^* D \mid L^* = D^* \mid L^* + \pi^* \bar{\varphi}_1^*(\varepsilon_1 L_1 \mid E_1) + \cdots$$
$$+ \pi^* \bar{\varphi}_{\ell-1}^*(\varepsilon_{\ell-1} L_{\ell-1} \mid E_{\ell-1}) + \pi^*(\varepsilon_\ell L_\ell \mid E_\ell) + \varepsilon L^* \mid L^*,$$

where $\bar{\varphi}_j = \varphi_j \mid E_\ell$. On the other hand, in view of $\psi_1^*(D) \mid L^* = \psi_1^*(D \mid p) \sim 0$, it follows that

$$-\varepsilon L^* \mid L^* \sim D^* \mid L^* + \pi^* \bar{\varphi}_1^*(\varepsilon_1 L_1 \mid E_1) + \cdots + \pi^*(\varepsilon_\ell L_\ell \mid E_\ell).$$

Hence, $S^\varepsilon(\mathcal{O}(N^\vee_{V_\ell/E_\ell})) \cong \mathcal{O}(D^* \mid L^*) \otimes \pi^* \mathcal{O}(\bar{\varphi}_1^*(\varepsilon_1 L_1 \mid E_1) \otimes \cdots \otimes \pi^* \mathcal{O}(\varepsilon_\ell L_\ell \mid E_\ell)$. Moreover,

$$\det S^\varepsilon(\mathcal{O}(N^\vee_{V_\ell/E_\ell})) = \mathcal{O}\left(\binom{\varepsilon-1+r}{\varepsilon-1}[\det N_{V_\ell/E_\ell}]\right)$$

and

$$\det\{\pi_* \mathcal{O}(D^* \mid L^*) \otimes \mathcal{L}\} \cong \det \pi_* \mathcal{O}(D^* \mid L^*)] \otimes \mathcal{L}^\alpha,$$

where $\mathcal{L}$ is the invertible sheaf and $\alpha = \binom{\varepsilon-1+r}{\varepsilon}$.

Hence,

$$-\binom{\varepsilon-1+r}{\varepsilon-1}[\det(N_{V_\ell/E_\ell})] \sim [\det \pi_* \mathcal{O}(D^* \mid L^*)] + \alpha \sum_{j=1}^{\ell} \bar{\varphi}_j^* \varepsilon_j L_j \mid E_j.$$

Thus,

$$\binom{\varepsilon-1+r}{\varepsilon-1}K(E_\ell) = \binom{\varepsilon-1+r}{\varepsilon-1}K(V_\ell) \mid E_\ell - [\det \pi_* \mathcal{O}(D^* \mid L^*)]$$

$$- \alpha \cdot \sum \varepsilon_j \bar{\varphi}_j^* (\varepsilon_j L_j \mid E_j)$$

$$= -[\det \pi_* \mathcal{O}(D^* \mid L^*)] + \binom{\varepsilon+r-1}{\varepsilon}\left\{\frac{\varepsilon}{r}\sum(v_j-1)\bar{\varphi}_j^*(L_j \mid E_j) - \varepsilon_j \bar{\varphi}_j^* L_j \mid E_j\right\}$$

$$\leqq -[\det \pi_* \mathcal{O}(D^* \mid L^*)] + \binom{\varepsilon+r-1}{\varepsilon}\left\{\sum \varepsilon_j \left(\frac{v_j-1-r}{r}\right)\bar{\varphi}_j^*(L_j \mid E_j)\right\},$$

because $L_j \mid E_j$ is effective and $\varepsilon_j \geqq \varepsilon$. If $v_j-1-r > 0$, then $L_j \mid E_j$ is exceptional. Hence

$$\varkappa\left(\sum \varepsilon_j \left(\frac{v_j-1-r}{r}\right)\bar{\varphi}_j^* L_j \mid E_j,\ E_\ell\right) \leqq 0.$$

On the other hand, the left hand side is not less than

$$\varkappa\left([\det \pi_*(\mathcal{O} D^*) \mid L^*] + \binom{\varepsilon-1+r}{\varepsilon-1}K(E_\ell),\ E_\ell\right).$$

From the following lemma, $\varkappa([\det(\pi_*(\mathcal{O} D^*) \mid L^*)], E_\ell) = n-r$, which implies $\varkappa(E_\ell) = -\infty$.

Lemma. Let M be a projective n-manifold and $\mathcal{E}$ a locally free sheaf of rank r. The projective bundle $\pi : \mathbb{P}(\mathcal{E}) \to M$ has the tautological line bundle E. Assume that $\varepsilon E + \pi^*(D)$ is very ample, where $\varepsilon > 0$ and D is a divisor on M. Then $\pi_* \mathcal{O}(\varepsilon E + \pi^* D)$ is the ample sheaf. Hence $[\det \pi_*(\mathcal{O}\, \varepsilon E + \pi^* D)]$ is ample.

Since $\mathbb{P}(S^\varepsilon(\mathcal{E})) \cong \mathbb{P}(S^\varepsilon(\mathcal{E}) \otimes \mathcal{O}(D)) \hookrightarrow \mathbb{P}(\varepsilon E + \pi^* D)$, whose isomorphism i transforms E_ε to $\varepsilon E + \pi_\varepsilon^* D$, we have the imbeddings $\mathbb{P}(\mathcal{E}) \subset \mathbb{P}(S^\varepsilon(\mathcal{E})) \hookrightarrow \mathbb{P}(H^0(\mathbb{P}, \mathcal{O}(\varepsilon E + \pi^* D)),$

E_ε being the tautological bundle of $\mathbb{P}(S^\varepsilon(\mathcal{E}))$. The hyperplane section H of $\mathbb{P}(H^0(\mathbb{P}, \mathcal{O}(\varepsilon E + \pi^*D)))$ induces $E_\varepsilon + \pi_\varepsilon^* D$ on $\mathbb{P}(S^\varepsilon(\mathcal{E}))$ and so $E_\varepsilon + \pi_\varepsilon^* D$ is ample. Hence $\pi_{\varepsilon *}\mathcal{O}(E_\varepsilon + \pi_\varepsilon^* D)$ $= \pi_* \mathcal{O}(\varepsilon E + \pi^* D) = S^\varepsilon(\mathcal{E}) \otimes \mathcal{O}(D)$ is also ample by Hartshorne's theorem.

Consequently we obtain

Theorem. Let V be a projective n-manifold of hyperbolic type, whose $|mK(V)|$ has no base point for $m \gg 0$. Then exceptional subvarieties for $\Phi_{mK(V)}$ are algebraic varieties of elliptic type.

Corollary. Let V be a submanifold of an abelian variety. Then the minimal canonical fibered manifold $f : V \to W$ is defined. K_V is a pull back of an ample divisor on W.

Problem. In the above situation, we assume that V is of hyperbolic type . Then is $|3K_V|$ very ample?

Remark. Let V be a 3-manifold of hyperbolic type whose $|mK(V)|$ has no base point for $m \gg 0$. We first assume that a non-singular surface S is the total exceptional set. Then we have two cases: (1) $f(S) = p \in W$. S^3 = the multiplicity of W at p is 2, 3, 4, ..., 9 since $-K(S)$ is ample. S is a rational surface called Del Pezzo surface, including $\mathbb{P}^2$, quadrics, cubics, ... (2) $f(S) = \Gamma$ a curve. $S \to \Gamma$ is a (2,1)-fibered surface whose general fiber is $\mathbb{P}^1$. The multiplicity of W at a general point of Γ is 2. Second we assume that a non-singular curve C is the total exceptional set of V. Then $C \cong \mathbb{P}^1$ and $N_{V/C} \cong \mathcal{O}(-1) \oplus \mathcal{O}(-1)$. Moreover, any non-singular curve can never be the total exceptional set of n-manifold ($n \geq 4$) of hyperbolic type under our assumption (i.e., $B_s|mK(V)| = \emptyset$).

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