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An example of unirational varieties in characteristic p.

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This note is to supplement a computation in my talk at Kyoto whose content can be found in [1].

Let X denote the Fermat variety of dimension 2r and of degree n, defined by the equation

(1) $$\sum_{i=1}^{2r+2} x_i^n = 0$$

in the projective space $\mathbb{P}^{2r+1}$ of characteristic $p \neq 2$. When $n \not\equiv 0 \pmod{p}$, X is a non-singular irreducible variety.
An irreducible variety (defined over a field k) is called unirational if its function field is contained in a purely transcendental extension of k. For simplicity, we assume k to be algebraically closed.

Proposition. Assume that $p^\nu \equiv -1 \pmod{n}$ for some integer ν. Then the Fermat variety X is unirational.

Proof. Put $q = p^\nu$ and $q + 1 = n \cdot a$ with some integer a.

Then the map

(2) $$(x_i) \longrightarrow (x_i^{\,a})$$

defines a surjective morphism of the Fermat variety of degree $q + 1$ onto that of degree n.

Hence it suffices to prove the case $n = q + 1$.

By a change of coordinates, we rewrite (1) as

(3) $$\sum_{i=1}^{r+1} (x_{2i-1}^{q+1} - x_{2i}^{q+1}) = 0.$$

Putting

(4) $$\begin{cases} y_{2i-1} = x_{2i-1} + x_{2i} \\ y_{2i} = x_{2i-1} - x_{2i} \end{cases} \qquad (1 \leq i \leq r+1),$$

we have

(5) $$\sum_{i=1}^{r+1} y_{2i-1} y_{2i} \; (y_{2i-1}^{q-1} + y_{2i}^{q-1}) = 0 \quad .$$

Setting $y_{2r+2} = 1$, we consider $y_1, \ldots, y_{2r+1}$ as the inhomogeneous coordinates on X. The function field $K = k(X)$ is given by

(6) $$K = k(y_1, \ldots , y_{2r+1}).$$

Put

(7) $$\begin{cases} y_2 = y_1 \cdot u \\ y_{2i} = y_{2i-1} \cdot t_{2i} \; u & (2 \leq i \leq r) \\ y_{2r+1} = u \; v \end{cases} \quad .$$

Then we have

(8) $$K = k(y_1, u, v, y_{2i-1}, t_{2i} \; (2 \leq i \leq r))$$

with the relation

(9) $y_1^{q+1}(1+u^{q-1})+\sum_{i=2}^{r} y_{2i-1}^{q+1}\; t_{2i}(1+u^{q-1}t_{2i}^{q-1})+v(u^{q-1}v^{q-1}+1)=0,$

i.e.

(10) $u^{q-1}(y_1^{q+1}+\sum_{i=2}^{r} y_{2i-1}^{q+1}\; t_{2i}^{q}\; + v^q) = -(y_1^{q+1}+\sum_{i=2}^{r} y_{2i-1}^{q+1}\; t_{2i}+v).$

Let

(11) $$\begin{cases} (y_{2i-1})^{1/q} = t_{2i-1} & (1\le i\le r)\\ t_2 = u(t_1^{q+1}+\sum_{i=2}^{r} t_{2i-1}^{q+1}\; t_{2i}+v). \end{cases}$$

Then the field

(12) $$\begin{aligned} K' &= k(t_1,\ u,\ v,\ t_{2i-1},\ t_{2i} \quad (2\le i\le r))\\ &= k(t_1,\ t_2,\ v,\ t_{2i-1},\ t_{2i} \quad (2\le i\le r)) \end{aligned}$$

is a purely inseparable extension of K, (8).

Now the relation (10) becomes

(13) $t_2^{q-1}(t_1^{q+1}+\sum_{i=2}^{r} t_{2i-1}^{q+1}\; t_{2i}+v) = -(t_1^{q(q+1)}+\sum_{i=1}^{r} t_{2i-1}^{q(q+1)}t_{2i}+v).$

This shows that v is a rational function of $t_1,\ldots,t_{2r}$, and hence

$$K' = k(t_1,\ t_2,\ \ldots,\ t_{2r})$$

is a purely transcendental extension of k of dimension $2r$.

This proves the unirationality of X, q.e.d.

Reference

[1] T. Shioda, An example of unirational surfaces in characteristic p, Math. Ann. (to appear).

[2] O. Zariski, On Castelnuovo's criterion of rationality $p_a = P_2 = 0$ of an algebraic surface, Illinois J. Math. 2 (1958) 303-315