

On blowing up algebraic cocycles

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§1. Blowing up algebraic cocycles

Let $f: \tilde{X} \rightarrow X$ be the blowing up of a non-singular algebraic variety X with non-singular center Y of codimension r . Then as for cohomology groups of \tilde{X} the following proposition is known.

Proposition 1. We have an isomorphism

$$\Phi: H^s(\tilde{X}, \mathbb{Z}) \cong H^s(X, \mathbb{Z}) \oplus H^{s-2}(Y, \mathbb{Z}) \oplus \cdots \oplus H^v(Y, \mathbb{Z}),$$

where \mathbb{Z} is the ring of integers and

$$v = \max \left(s - 2r + 2, \frac{1}{2}(1 + (-1)^{s-1}) \right).$$

Here the isomorphism is given as follows:

Letting $\tilde{Y} = f^{-1}(Y)$ be the inverse image of Y , $j: \tilde{Y} \hookrightarrow \tilde{X}$ and $g: \tilde{Y} \rightarrow Y$ the injection map and the restriction of f to \tilde{Y} respectively,

and $\hat{\eta}$ the cohomology class dual to the cycle \tilde{Y} , we define the homomorphism Φ^{-1} from $H^s(X, \mathbb{Z}) \oplus \sum_{\alpha \geq 1} H^{s-2\alpha}(Y, \mathbb{Z})$ to $H^s(\tilde{X}, \mathbb{Z})$ by

$$\Phi^{-1}\left(x + \sum_{\alpha \geq 1} u_{s-2\alpha}\right) = f^*(x) + \sum j_*(g^*(u_{s-2\alpha}) \cdot \hat{\eta}^{\alpha-1}),$$

where j_* is the Gysin homomorphism induced from the injection j .

Now we consider an algebraic cocycle z dual to an irreducible subvariety Z of X of codimension s which contains Y properly. Let \tilde{Z} be the strict transform of Z by the blowing up f and \tilde{z} the cohomology class dual to \tilde{Z} . Then we have

Proposition 2. Every component of $\Phi(\tilde{z})$ is an algebraic cocycle.

A proof of this proposition is given as follows. Let $\nu(Y, X)$ be the normal bundle of Y in X . Then $g: \tilde{Y} \rightarrow Y$ is a fibre bundle associated to $\nu(Y, X)$ with the projective space \mathbb{P}^{r-1} as fibre and hence there is a canonically defined complex line bundle L on \tilde{Y} such that

its restriction to a fibre is associated to a hyperplane of \mathbb{P}^{r-1} . We denote by $h = c_1(L)$ the first Chern class of L . Putting

$$\Phi(\tilde{Z}) = f^*(z) + \sum j_*(g^*(u_{2(s-d)})) \cdot \tilde{y}^{\alpha-1},$$

we apply the homomorphism j^* on both hands.

Since $j^*(\tilde{Z})$ is a linear combination of duals of irreducible components of $\tilde{Z} \cap \tilde{Y}$ and

$$j^*(j_*(g^*(u_{2(s-d)})) \cdot \tilde{y}^{\alpha-1}) = g^*(u_{2(s-d)}) \cdot (-h)^\alpha,$$

Grothendieck [1] yields analyticity of $u_{2(s-d)}$.

It is easily seen that $X = Z$, which ends the proof.

When Z is non-singular, we have the following

Proposition 3.

$$\tilde{Z} \equiv f^*(z) + \sum_{\alpha \geq 1} (-1)^\alpha j_*(g^*(c_{s-\alpha}(\nu(Z, X)))) \cdot \tilde{y}^{\alpha-1} \pmod{p},$$

where $\nu(Z, X)$ is the restriction to Y of the normal bundle of Z in X and p is a prime.

Proof is omitted.

§2 Analyticity of Steenrod powers of algebraic cocycles

Letting $P^k: H^s(X, \mathbb{Z}_p) \rightarrow H^{s+2k(p-1)}(X, \mathbb{Z}_p)$ be the Steenrod power operation, we consider the Steenrod power $P^k(z)$ of an algebraic co-cycle z dual to a subvariety Z . When Z is non-singular, we obtain from Thom [3,4] the following proposition.

Proposition 4. Let $\nu(Z)$ be the normal bundle of Z in X and λ_* the Gysin homomorphism induced by the injection $\lambda: Z \hookrightarrow X$. Then we have

$$P^k(z) = \lambda_*(T^k(\nu(Z))),$$

where the cohomology class $T^k(\nu(Z))$ is a polynomial of Chern class $c_e(\nu(Z))$ of $\nu(Z)$ defined as follows: Let $\sum c_e(\nu(Z))x^e = \prod (1 + t_e x)$ be formal factorization, and

$$T^k(\nu(Z)) = \sum_{1 \leq d_1 < d_2 < \dots < d_k \leq s} (t_{d_1} t_{d_2} \dots t_{d_k})^{p-1}.$$

If Z has singularities, by a theorem of Hironaka [2] we can resolve the singularities by successive applications of blowings-up. Now we consider the blowing-up of X with

non-singular center Υ contained in Z and we use the same notations as in §1. Since $P^k(z) = f_* f^*(P^k(z)) = f_* P^k(f^*(z))$, f_* being the Gysin homomorphism induced by f , analyticity of $P^k(f^*(z))$ induces analyticity of $P^k(z)$. By means of Thom isomorphism we obtain

Lemma. $P^k j_*(w) = j_*(P^k w) + j_*(P^{k-1} w) \cdot \bar{y}^{p-1}$
for $w \in H^*(\tilde{\Upsilon}, \mathbb{Z}_p)$.

From this Lemma and Proposition 2 we infer readily analyticity of $P^k(f^*(z))$ by induction on singularities and codimension of subvariety. Thus we obtain

Theorem. Steered powers of analytic cocycles are also analytic.

References

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