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Lower bounds of Growth order of solutions of Schrödinger equations with homogeneous potentials

Jun Uchiyama

Kyoto University of
Industrial Arts and
Textile Fibers

We shall study on the asymptotic behavior as $|x|$ tends to ∞ of the solution $u(x) \in H^2_{loc}(\Omega)$ satisfying the following equation.

(1) $\quad -\Delta u(x) + q(x)u(x) = \lambda u(x) \quad$ for $x \in \Omega \subset R^n (n \geqslant 3)$,

where λ is a positive constant and $\Omega \supset E_{R_0} = \{ x \in R^n ; |x| > R_0 \}$.

The conditions to be imposed on the potential $q(x)$ are following ones.

Assumption 1. The potential $q(x)$ is real valued and satisfies the following inequality:

(2) $\quad |x| < Dq, \tilde{x} > \leqslant (-2\gamma) q(x) \quad$ for $x \in \Omega$,

where γ is some constant, Dq is a gradient of $q(x)$, $\tilde{x} = \frac{x}{|x|}$, and $< \cdot , \cdot >$ denotes the inner product of C^n.

Assumption 2. For $q(x)$, we have

(3) $$\sup_{x \in R^n} \int_{|x-y| \leqslant 1} \frac{|q(y)|^2}{|x-y|^{n-4+\mu}} \, dy < +\infty$$

for some constant $\mu > 0$.

Assumption 3. The unique continuation property holds.

Remark. If $q(x)$ is a homogeneous functions of x of degree -2γ, then (2) is satisfied. So we cannot expect that $q(x)$ decays uniformaly at infinity.

Example. The potential of the Schrödinger operator of an atom (or ion) consisting of a nucleus with charge +Z and m electrons given by

$$(4)\qquad q(x) = -\sum_{k=1}^{m} \frac{Z}{r_k} + \sum_{1\leq k<l\leq m} \frac{1}{r_{kl}},$$

where

$$r_k^2 = \sum_{l=0}^{2} |x_{3k-l}|^2,\quad r_{kj}^2 = \sum_{l=0}^{2} |x_{3k-l}-x_{3j-l}|^2,$$ is a

homogeneous function of $x \in R^{3m}$ of degree -1(i.e. $\gamma = \frac{1}{2}$)

We can admit u(x) be a real valued function. Our aim is to take α in the following estimation (5) for the not identically vanishing solution u(x) of (1) as large as possible.

$$(5)\qquad \liminf_{r\to\infty} r^{-\alpha}\int_{S_r} (\text{some form of } u)dS > 0,$$

where $S_r = \{ x \in R^n;\ |x| = r \}$.

To this end we introduce v(x) by $v(x) = |x|^{\frac{n-1}{2}} u(x)$, and then v(x) satisfies the following equation

$$(6)\qquad -\Delta v + \frac{n-1}{|x|}\langle Dv, \tilde{x}\rangle + (q(x) + \tilde{q}(x) - \lambda)v(x) = 0 \text{ in } \Omega,$$

where

$$(7)\qquad \tilde{q}(x) = \frac{(n-1)(n-3)}{4|x|^2}.$$

According to the caluculation used in Ikebe-Uchiyama [4], we have by (6) and integration by parts

$$(8)\qquad 0 = \int_{B_{sr}} \{\text{the left side of (6)}\}\{2|x|^{\alpha}\langle Dv, \tilde{x}\rangle - k|x|^{\alpha-1}v\}dx$$

$$= \left(\int_{S_r} - \int_{S_s}\right)\left\{\text{some form of } v\right\} dS + \int_{B_{sr}}\left\{\text{another form of } v\right\} dx,$$

where $B_{sr} = \left\{ x \in R^n;\ s < |x| < r \right\} \subset \Omega$. Here parameter k is introduced according to the method used in Agmon [1]. We define the surface integral on S_r in (8) as $F(r, \alpha; k)$. Namely

$$(9) \quad F(r,\alpha;k) = \int_{S_r} \Big\{ 2\langle Dv, \tilde{x}\rangle^2 - |Dv|^2 + (\lambda - q(x) - \tilde{q}(x))|v|^2 - \frac{k}{|x|}\langle Dv, \tilde{x}\rangle v + \frac{(\alpha+n-2)}{2}\frac{k}{|x|^2}|v|^2 \Big\} |x|^{\alpha} dS.$$

Then let us rewrite (8), and we have

$$(10) \quad F(r,\alpha;k) - F(s,\alpha;k) = \int_{B_{sr}} \Big[(3-\alpha-n-k)(|Dv|^2 - \langle Dv, \tilde{x}\rangle^2) + (\alpha+n-k-1)\langle Dv, \tilde{x}\rangle^2 + \Big\{ (\alpha+n+k-1)\lambda - (\alpha+n+k-1)q(x) - |x|\langle Dq, \tilde{x}\rangle - (\alpha+n+k-3)\tilde{q}(x) + \frac{k}{2}(\alpha+n-2)(\alpha+n-3)\frac{1}{|x|^2}\Big\} |v|^2 \Big] |x|^{\alpha-1} dx, \text{ for } r > s > 0,$$

where $B_{sr} \subset \Omega$.

By (10) we have

<u>Lemma 1.</u> If $0 \leq \gamma \leq 1$, we have for any $r > s > 0$ satisfying $B_{sr} \subset \Omega$

$$F(r, \alpha_0; \gamma) \geqslant F(s, \alpha_0; \gamma),$$

where $\alpha_0 = 1-n+\gamma$.

Noting the following relation

$$(11) \quad F(r, \alpha_0; \gamma) = r^{\gamma} \int_{S_r} \Big[2\langle Du, \tilde{x}\rangle^2 - |Du|^2 + \frac{(n-1-\gamma)}{|x|}\langle Du, \tilde{x}\rangle u + (\lambda - q(x))|u|^2 +$$

$$+ \frac{(1-\gamma)(n-1-\gamma)}{2|x|^2} |u|^2 \Big] dS$$

and $u(x) \in H^2_{loc}(\Omega)$, we have

<u>Lemma 2.</u> If $\Omega = R^n$ and $0 < \gamma \leq 1$, we have

$$\lim_{s \to 0} F(s, \alpha_0; \gamma) = 0.$$

Noting that $u(x)$ is a not identically vanishing solution of (1), we have by (10) and Lemma 2

<u>Lemma 3.</u> If $\Omega = R^n$ and $0 < \gamma \leq 1$, there exists some $r_0 > 0$ such that

$$F(r_0, \alpha_0; \gamma) > 0.$$

So by Lemma 1 and Lemma 3, we have

<u>Theorem 1.</u> If $\Omega = R^n$, and if u is a not identically vanishing solution of (1), then we have

$$\liminf_{R \to \infty} R^{\gamma - 1} \int_{|x| \leq R} |u(x)|^2 dx > 0, \text{ when } 0 < \gamma < 1,$$

and

$$\liminf_{R \to \infty} (\log R)^{-1} \int_{|x| \leq R} |u(x)|^2 dx > 0, \text{ when } \gamma = 1.$$

Hereafter we only assume $\Omega \supset E_{R_0}$. In order to prove Lemma 3, we introduce the function $w_m(x)$ by

(12) $$w_m(x) = e^{mf(|x|)} v(x),$$

and the form $F(r, \alpha, \beta; m; k; f)$ by

(13) $$F(r, \alpha, \beta, m; k; f(r)) = \int_{S_r} \Big[2 \langle Dw_m, \tilde{x} \rangle^2 - |Dw_m|^2 -$$

$$- \frac{k}{|x|} \langle Dw_m, \tilde{x} \rangle w_m + \Big\{ \lambda - q(x) - \tilde{q}(x) + m^2 f'^2 - mf'' -$$

$$- |x|^{\beta-\alpha} + \frac{k}{2}(\alpha+n-2)\frac{1}{|x|^2} + \frac{km}{|x|}f'\Big\}|w_m|^2\Big]|x|^{\alpha}dS$$

$$= e^{2mf(r)}\Big[F(r,\alpha;k) + \int_{S_r}\Big\{2mf'\langle Dv,\tilde{x}\rangle v +$$

$$+ (2m^2f'^2 - mf'' - |x|^{\beta-\alpha})|v|^2\Big\}|x|^{\alpha}dS\Big].$$

In order to show $F(r,\alpha_0;\gamma) > 0$, it is sufficient to prove $F(r,\alpha_0,\beta,m;\gamma;f(r)) > 0$ and $\int_{S_r}\Big\{2mf'\langle Dv,\tilde{x}\rangle v +$

$+ (2m^2f'^2 - mf'' - |x|^{\beta-\alpha})|v|^2\Big\}|x|^{\alpha_0}dS < 0$ for some β, m and f(r).

To investigate the property of $F(r,\alpha,\beta,m;k;f(r))$, we apply the caluculation, which is similar to (8) or (10), to $F(r,\alpha,\beta,m;k;f(r))$, noting the identy

(14) $$2\int_{B_{sr}}\langle Dw,\tilde{x}\rangle w|x|^{\beta}dx = \int_{S_r}|w|^2|x|^{\beta}dS - \int_{S_s}|w|^2|x|^{\beta}dS -$$

$$- (n+\beta-1)\int_{B_{sr}}|w|^2|x|^{\beta-1}dx.$$

So we have

(15) $$F(r,\alpha,\beta,m;k;f) - F(s,\alpha,\beta,m;k;f) =$$

$$= \int_{B_{sr}}\Big[(3-\alpha-n-k)(|Dw_m|^2 - \langle Dw_m,\tilde{x}\rangle^2) +$$

$$+ (\alpha+n-k-1+4m|x|f')\langle Dw_m,\tilde{x}\rangle^2 - 2|x|^{\beta-\alpha+1}\langle Dw_m,\tilde{x}\rangle w_m +$$

$$+ \Big\{(\alpha+n+k-1)\lambda - (\alpha+n+k-1)q(x) - |x|\langle Dq,\tilde{x}\rangle -$$

$$- (\alpha+n+k-3)\tilde{q}(x) + (\alpha+n+k-1)m^2f'^2 + 2m^2|x|f'f'' -$$

$$- (\alpha+n-1)mf'' + k(\alpha+n-2)\frac{1}{|x|}mf' - |x|mf''' -$$

$$- (\beta+n-1)|x|^{\beta-\alpha} + \frac{k}{2}(\alpha+n-2)(\alpha+n-3)\frac{1}{|x|^2}\Big\}|w_m|^2\Big]|x|^{\alpha-1}dx.$$

Here we put $\alpha = \alpha_0$, $\beta = \alpha_0 - \delta$, $k = \gamma$ and $f(r) = r^\varepsilon$. Then we have

<u>Lemma 4.</u> If $\varepsilon \geqslant 2(1-\delta)$, $\varepsilon > 0$, $\varepsilon > 1-\gamma$ and $\delta > 0$, then there exist some $R_1 > R_0$ and some $m_0 > 0$ such that for any $r > s \geqslant R_1$ and for any $m \geqslant m_0$ we have

$$F(r, \alpha_0, \alpha_0 - \delta, m; \gamma; r^\varepsilon) \geqslant F(s, \alpha_0, \alpha_0 - \delta, m; \gamma; s^\varepsilon).$$

Unique continuation property leads that there exists some sequence $\{r_l\}_{l=1,2,\ldots}$ such that $R_1 < r_1 < r_2 < \cdots < r_l < r_{l+1} < \cdots$, $\lim_{l\to\infty} r_l = \infty$ and $\int_{S_{r_l}} |v|^2 dS > 0$ $(l=1,2,\ldots)$.

Since in the right hand term of (13) the coefficient of e^{2mr^ε} is a quadratic form of m, in which the coefficient of m^2 is positive, we have

<u>Lemma 5.</u> There exists some constant $m_1 \geqslant m_0$ such that

$$F(r_1, \alpha_0, \alpha_0 - \delta, m_1; \gamma; r_1^\varepsilon) > 0.$$

By Lemma 4 and Lemma 5, we have

<u>Lemma 6.</u> Under the same conditions imposed in Lemma 4, we have

$$F(r, \alpha_0, \alpha_0 - \delta, m_1; \gamma; r^\varepsilon) > 0 \text{ for any } r \geqslant r_1.$$

Now the problem we consider is reduced to the following Case I or Case II:

<u>Case I.</u> There exists some sequence $\{r'_l\}_{l=1,2,\ldots}$ such that

$$r_1 < r'_1 < r'_2 < \cdots < r'_l < r'_{l+1} < \cdots, \quad \lim_{l\to\infty} r'_l = \infty$$

and $\int_{S_{r'_l}} \langle Dv, \tilde{x} \rangle v dS \leqslant 0$ $(l=1,2,\ldots)$,

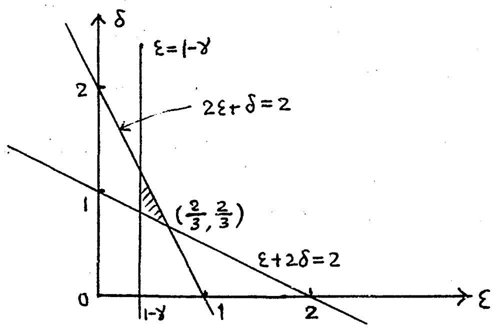
<u>Case II.</u> There exists some $R_2 \geqslant r_1$ such that $\int_{S_r} \langle Dv, \tilde{x} \rangle v dS > 0$

(i.e. $\frac{d}{dr}\int_{S_r} |u|^2 dS > 0$) for any $r \geqslant R_2$.

By (13), we have

<u>Lemma 7.</u> We assume the same conditions imposed in Lemma 4. Moreover if Case I holds and if $2\varepsilon - 2 < -\delta$, there exists some $R_3 \geqslant r_1$ such that $F(R_3, \alpha_0; \gamma) > 0$.

<u>Lemma 8.</u> If $\frac{1}{3} < \gamma \leqslant 1$, and if Case I holds, then the statement of Lemma 7 holds.



If Case II holds, $\int_{S_r} |u|^2 dS$ is a monotone increasing function.

Consequently under either Case I or Case II, we have

<u>Theorem 2.</u> If $\Omega \supset E_{R_0}$, and if u is a not identically vanishing solution of (1), then we have

$$\liminf_{R \to \infty} R^{\gamma - 1} \int_{R_0 \leqslant |x| \leqslant R} |u(x)|^2 dx > 0, \quad \text{when } \frac{1}{3} < \gamma < 1,$$

and

$$\liminf_{R \to \infty} (\log R)^{-1} \int_{R_0 \leqslant |x| \leqslant R} |u(x)|^2 dx > 0, \quad \text{when } \gamma = 1.$$

<u>Corollary.</u> If $\Omega = R^n$ and $0 < \gamma \leqslant 1$, or if $\Omega \supset E_{R_0}$ and $\frac{1}{3} < \gamma \leqslant 1$, the Schrödinger operator appearing on the left side of (1) has no positive eigenvalues.

Remark. Weidmann [5] has (not explicitly) shown under the conditions $\Omega = R^n$ and $0 < \gamma \leq \frac{1}{2}$,

$$\liminf_{R \to \infty} R^{2\gamma - 1} \int_{|x| \leq R} |u(x)|^2 dx > 0, \quad (0 < \gamma < \tfrac{1}{2})$$

and

$$\liminf_{R \to \infty} (\log R)^{-1} \int_{|x| \leq R} |u(x)|^2 dx > 0, \quad (\gamma = \tfrac{1}{2}).$$

It has shown that the Schrödinger operator appearing in (1) has no positive eigenvalue by Weidmann [5] under the conditions $\Omega = R^n$ and $0 < \gamma \leq \frac{1}{2}$, Weidmann [6] under the conditions $\Omega = R^n$ and $\frac{1}{2} \leq \gamma < 1$ and Agmon [2], [3] under $\Omega \supset E_{R_0}$ and $\frac{1}{2} \leq \gamma < 1$.

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