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The limiting absorption principle for Dirac operators

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In the present paper we are concerned with the Dirac operator

$$L = -i \sum_{j=1}^{3} \alpha_j \frac{\partial}{\partial x_j} + \beta + Q(x) \quad (x \in R^3) ,$$

which appeares in relativistic quantum mechanics. The matrices α_j and β (called the Dirac matrices) are 4×4 Hermitian matrices with the anti-commutation relation

$$\alpha_j \alpha_k + \alpha_k \alpha_j = 2 \delta_{jk} I , \ j,k = 1,2,3,4$$

($\alpha_4 = \beta$, I is the unit matrix). The potential $Q(x)$ is a 4×4 Hermitian matrix-valued function.

The unperturbed operator L_0 (as $Q(x) \equiv 0$) defined on $\mathcal{C}_0^\infty = \left[C_0^\infty(R^3) \right]^4$ is essentially selfadjoint in $\mathcal{L}^2 = \left[L^2(R^3) \right]^4$, that is, the closure $H_0 = (L_0)^{**}$ is the unique selfadjoint extension of L_0 . Then it turns out that the domain of H_0

$$D(H_0) = \mathcal{H}^1 = \left[H^1(R^3) \right]^4$$

($H^1(R^3) = \left\{ u(x) \in L^2(R^3) \;;\; \frac{\partial}{\partial x_j} u(x) \in L^2(R^3)\;,\; j=1,2,3 \right\}$,

where the derivatives are taken in the distribution sense), and the essential spectrum

$$\sigma_e(H_0) = (-1,\, 1)^c$$

(the complement of the interval $(-1, 1)$ in the real line).

Proposition 1. Let $Q(x)$ satisfy

(1) $$|Q(x)| \longrightarrow 0 \quad (|x| \to \infty)$$

and

(2) $$\sup_{x \in R^3} \int_{|x-y| \leq 1} |Q(y)|^2 \; |x-y|^{-1-\delta} \, dy < +\infty$$

for some $\delta > 0$, where $|M|$ for a matrix M indicates the square root of the maximum eigenvalue of $M^* M$. Then $L = L_0 + Q(x)$ has the unique selfadjoint realization $H = H_0 + Q$ with the domain $D(H) = \mathcal{H}^1$ and the essential spectrum $\sigma_e(H) = (-1, 1)^c$.

For the proof of the above proposition see, e.g., Jörgens [1].

Remark 1. Coulomb potentials do not fulfill the condition (2). But the above result holds also by replacing (2) by a condition

(3) $$\begin{cases} |Q(x)| \leq \dfrac{c_1}{|x|} & (|x| \leq 1),\quad \frac{1}{2} > c_1 > 0 \\ |Q(x)| \leq c_2 & (|x| \geq 1),\quad c_2 > 0 \end{cases}$$

(see Jörgen [1] and Arai [2]) .

There are many works related to the spectral and scattering theory for the Dirac operator (e.g., Birman [3], Titchmarsh [4], Prosser [5], Roze [6], Evans [7], Thompson [8], Mochizuki [9], Eckardt [10], [11]). Prosser [5] shows that the wave operator

$$W_{\pm} = \operatorname*{s-lim}_{t \to \pm\infty} \exp(itH) \exp(-itH_0)$$

exists under the main assumption

(4) $$|Q(x)| = O(|x|^{-1-h}) \qquad (h > 0)$$

at infinity, and that the scattering operator

$$S = W_+^* W_-$$

is unitary for a class of potentials with compact support. Eckardt [10] proves the existence of wave operators under a weaker condition

$$|Q(x)| (1 + |x|)^{-1/2 + \delta} \in L^2(R^3) \qquad (\delta > 0).$$

In [12] and [13] we assume (4) and

$$Q(x) \in \mathcal{B}^1(R^3)$$

(i.e., every component of $Q(x)$ is bounded and has bounded continuous first derivatives). We can assume some local singularities of $Q(x)$, but for the sake of simplicity we omit them. In [12] we show that the limiting absorption principle holds on $[-1, 1]^c$. The limiting absorption principle is, roughly speaking, to investigate the resolvent of H

near the spectrum. Let $R(z) = (H - z)^{-1}$ be the resolvent of H for non-real z. As z tends to the spectrum, the limit of $R(z) f$ for $f \in \mathcal{L}^2$ does not exist generally in $\mathcal{L}^2$. The limit of $R(z) f$, however, exists for appropriate functions $f(x)$ in some weighted Hilbert spaces (the method is called the limiting absorption principle). We introduce two weighted functional spaces

$$\mathcal{L}^2_t = \left\{ u \; ; \; \int_{R^3} (1 + |x|)^{2t} \; |u(x)|^2 \, dx < +\infty \right\},$$

$$\mathcal{H}^1_{-t} = \left\{ u \; ; \; \int_{R^3} (1 + |x|)^{-2t} \left(|u(x)|^2 + \sum_{j=1}^{3} \left| \frac{\partial}{\partial x_j} u(x) \right|^2 \right) dx < +\infty \right\}.$$

Theorem 1. (the limiting absorption principle) Let $t > 1/2$. Then for every real λ such that $|\lambda| > 1$, there exist bounded operators $R^+(\lambda)$, $R^-(\lambda)$ on $\mathcal{H}^1_{-t}$ to $\mathcal{L}^2_t$ such that

$$\operatorname*{s-lim}_{z \to \lambda \pm 0i} R(z) f = R^{\pm}(\lambda) f \qquad \text{in } \mathcal{H}^1_{-t}$$

for $f \in \mathcal{L}^2_t$. For every $f \in \mathcal{L}^2_t$, $R(z) f$ is strongly continuous in the topology of $\mathcal{H}^1_{-t}$ with respect to z with the boundary values $R^+(\lambda) f$, $R^-(\lambda) f$.

The following assertion follows directly from Theorem 1.

Corollary 1 . $[-1, 1]^c$ is absolutely continuous spectrum of H .

In [13] we see eigenfunction expansions and scattering theory under the same condition as in [12].

We shall summarize the results in [13].

There exist 4×4 matrix-valued functions $\Phi^{\pm}_{\nu}(x,r)$ $(\nu = p, n)$ for $x \in R^3$ and $r > 0$. Every component of $\Phi^{\pm}_{\nu}(x,r)$ is a $L^2(S)$-valued function, locally Hölder continuous in $L^2(S)$ with respect to x and locally bounded in $L^2(S)$ with respect to r (S is the unit surface about the origin). $\Phi^{\pm}_{\nu}(x,r)$ might be called generalized eigenfunctions in the following sense :

$$(L_0 + Q(x)) \int_S (\Phi^{\pm}_{\nu}(x,r))(\omega)\ h(\omega)\ d\omega = \tau_{\nu} \sqrt{r^2 + 1} \int_S (\Phi^{\pm}_{\nu}(x,r))(\omega)\ h(\omega)\ d\omega$$

for any $h \in \mathcal{L}^2(S) = (L^2(S))^4$, where $\tau_p = 1$, $\tau_n = -1$. Let

$$(Z^{\pm}_{\nu} f)(r\cdot) = (2\pi)^{-3/2}\ \text{l.i.m.} \int_{R^3} \Phi^{\pm}_{\nu}(x,r)^* f(x)\ dx$$

for $f \in \mathcal{L}^2$. Then $Z^{\pm}_{\nu}$ is a partially isometric operator in $\mathcal{L}^2$ with the initial set $(I - E(1)^{1)}) \mathcal{L}^2$ $(\nu = p)$, $E(-1-0) \mathcal{L}^2$ $(\nu = n)$

For $f \in \mathcal{L}^2$

1) $E(\cdot)$ is the right-continuous resolution of the identity associated with H .

$$\|f\|^2 = \|Z_p^{\pm} f\|^2 + \|Z_n^{\pm} f\|^2 + \sum_j |(f, \varphi_j)|^2 ,$$

where $\{\varphi_j\}$ is the set of the orthonormalized eigenfunctions for the discrete eigenvalues in $[-1, 1]$ (it may be empty). We can construct the stationary wave operator

$$U_{\pm} f = (Z_p^{\pm})^*(\hat{f}) + (Z_n^{\pm})^*(\hat{f})$$

isometric from $\mathcal{L}^2$ onto $(I - E(1) + E(-1-0))\mathcal{L}^2$, where $\hat{f}$ is the Fourier image of f . Then we have

$$U_{\pm} = W_{\pm}$$

(that is, the above stationary wave operator coincides with the time-dependent wave operator $W_{\pm}$), and that the scattering operator $S = W_+^* W_-$ is unitary.

We shall now give the result for the long range potential. The potential $Q(x)$ is assumed to satisfy the following condition (A) :

(A) each component of $Q(x)$ is continuously differentiable exept at a finite number of singularities, satisfying (2) or (3), and

(4) $$|Q(x)| = O(|x|^{-\delta})$$

(5) $$\sum_{j=1}^{3} \left| \frac{\partial}{\partial x_j} Q(x) \right| = O(|x|^{-1-\delta})$$

at infinity for some $\delta > 0$.

Theorem 2. Let $Q(x)$ satisfy the condition (A). Then the number of eigenvalues of $H = H_0 + Q$ is, if it exists, is at most finite in $[-1-\delta, 1+\delta]^c$ for every $\delta > 0$. $\{\lambda_n\}$ denotes the set of eigenvalues in $[-1, 1]^c$ (it may be empty). Then each λ_n is of finite multiplicity. $[-1, 1]^c - \{\lambda_n\}$ is the absolutely continuous spectrum of $H = H_0 + Q$.

Proof. Let us take any $f \in \mathcal{H}^1_t$ ($t > 1/2$) and non-real z. Then $u = (H - z)^{-1} f$ fulfills

$$(L_0 + Q(x)) u(x) - z u(x) = f(x)$$

and, by $L_0^2 = (-\Delta + 1) I$ (which is easily checked by the anti-commutation relation of α_j),

$$-\Delta u(x) + L_0 (Q(x) u(x)) + z Q(x) u(x) - (z^2 - 1) u(x) = L_0 f(x) + z f(x) .$$

In view of this fact we notice that a result on Schrödinger operators with long range potentials, obtained by Ikebe-Saito [14], will be applicable to our assertion. The above theorem is proved along the same line of Ikebe-Saito [14].

A sufficient condition for the non-existence of the eigenvalues in $[-1, 1]^c$ is given as follows.

Theorem 3. Assume that

$$L = L_0 + Q(x)$$
$$\equiv -i \sum_{j=1}^{3} \alpha_j \left(\frac{\partial}{\partial x_j} + i A_j(x) \right) + \beta + q(x) I$$
$$\left(Q(x) \equiv \sum_j A_j(x) \alpha_j + q(x) I \right)$$

such that the scalar potential $A_j(x)$ and $q(x)$ belong to $C^1(R^3 - \mathcal{E})$ ($\mathcal{E}$ is a set of a finite number of points), satisfying

$$\sum_{j=1}^{3} \left(\left| A_j(x) \right| + |x| \left| \operatorname{grad} A_j(x) \right| \right)$$
$$+ \left| q(x) \right| + |x| \left| \operatorname{grad} q(x) \right| = o(1) \quad (|x| \to \infty).$$

Then the selfadjoint extension H has no $\mathcal{L}^2$-eigenfunctions in $\left[-1, 1 \right]^c$.

Proof. Let $H u = \lambda u$ ($|\lambda| > 1$, $u \in \mathcal{L}^2$). Then u is a solution of a Schrödinger equation

$$-\Delta u(x) - 2i \sum_{j=1}^{3} A_j(x) \frac{\partial u}{\partial x_j} - i \sum_{j=1}^{3} \frac{\partial q}{\partial x_j} \alpha_j u(x)$$
$$-i \sum_{j,k=1}^{3} \frac{\partial A_j}{\partial x_k} \alpha_k \alpha_j u(x) + \left(\sum_{j=1}^{3} A_j(x)^2 + 2\lambda q(x) \right.$$
$$\left. - q(x)^2 \right) u(x) = (\lambda^2 - 1) u(x) .$$

Then we can show $u = 0$ by the method in Ikebe-Saito [14], Remark on the proof of Lemma 2.5.

Remark 2. When the potential $Q(x)$ is a spherically symmetric scalar function, spectral problems for Dirac operators frequently reduces, by separation of variables, to investigate 2×2 differential operators

$$h_k=\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\frac{d}{dr}+\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}-\frac{k}{r}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}+q(r)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$0<r<\infty \quad (k = \pm 1, \pm 2, \pm 3, \ldots) ,$$

where $r = |x|$ and $q(r) = Q(x)$ (see, e g., Dirac [15]). The operator h_k is also studied by many authors (e.g., Titchmarsh [16], Weidmann [17]). Weidmann [16] shows that every selfadjoint realization A_k of h_k has the essential spectrum $(-1, 1)^c$, and that the spectrum of A_k is absolutely continuous in $[-1, 1]^c$, when $q(r) = \frac{c}{r}$ (c is an arbitrary real number).

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