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コホモロジー複素射影空間上の余次元1の軌道を持ったコンパクト変換群について

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序.　$2n$ 次元の連結可微分閉多様体 M は, 有理数係数のコホモロジー環が $P_n(C)$ のコホモロジー環と同型であるとき, n 次元有理コホモロジー複素射影空間であるという。この報告において, 単連結な n 次元有理コホモロジー複素射影空間 M 上の, コンパクト連結リイ群 G の可微分作用が $2n-1$ 次元の軌道 G/K を持つ場合について考察する。可微分スライス定理を使って, G/K が主軌道であり, M は丁度2つの特異軌道 G/K_1, G/K_2 を持つことが分かり, さらに,

$$\rho_s : K_s \longrightarrow O(k_s), \quad s=1,2$$

をスライス表現とするとき, M はG多様体として

$$(*) \qquad G \times_{K_1} D^{k_1} \underset{k}{\cup} G \times_{K_2} D^{k_2}$$

と, 同変微分同相になることが分かる。今, $u \in H^2(M;Q)$

を生成元とし，$f_s : G/K_s \longrightarrow M$ を包含写像とする。整数 n_s $(s=1.2)$ を

$$f_s^*(u^{n_s}) \neq 0, \quad f_s^*(u^{n_s+1}) = 0$$

によって定めると，

$$n = n_1 + n_2 + 1,$$

$$2 \leqq k_s \leqq 2(n-n_s), \quad (s=1.2)$$

の成り立つことが分かる。更に，ポアンカレ双対定理，トム同型定理，その他のコホモロジー論的手段を駆使して，次の定理を得る。

定理 0.

(A) G/K_1, G/K_2 が共に向きづけ可能であるとき.

(i) $k_1 \equiv k_2 \pmod 2$ ならば．G/K_s は n_s 次元有理コホモロジー複素射影空間になり，$k_s = 2(n-n_s)$ が成り立つ.

(ii) $k_1 \equiv 0 \pmod 2$, $k_2 \equiv 1 \pmod 2$ ならば．$k_1 + k_2 = n+2$ が成り立ち，次の２つの場合が起る：

(a) $n_1 = n_2$ 且つ

$$P(G/K_1) = (1+t^{k_2-1})(1+t^2+\cdots+t^{2n_1}),$$

$$P(G/K_2) = (1+t^{k_1-1})(1+t^2+\cdots+t^{2n_2}).$$

(b) $k_1 = 2(n_2+1)$, $k_2 = n_1 - n_2 + 1$ 且つ

$$P(G/K_1) = (1+t^{n_1-n_2})(1+t^2+\cdots+t^{n_1+n_2}),$$

$$P(G/K_2) = (1+t^{n})(1+t^2+\cdots+t^{2n_2}).$$

(B) G/K_1 が向きづけ可能で，G/K_2 が向きづけ不能のとき．$n\equiv 0 \pmod 2$ となり，G/K_1 は $n-1$ 次元有理コホモロジー複素射影空間で，G/K_2^0 は n 次元有理コホモロジー球面である．

(C) G/K_1，G/K_2 が共に向きづけ不能であれば，$n=3$ となり，$s=1,2$ に対して

$$P(G/K_s)=1+t^2,\ P(G/K_s^0)=(1+t^2)^2$$

が成り立つ。

§1 において，若干の一般論を展開し，§2 において，定理0の詳細な証明を行う。 §3 において，定理0の夫々の場合に対応する例を挙げる。 分類定理を完成するには，定理0の条件を満たす等質空間 G/K_s を数え挙げ，夫々について，ある条件を満たす表現 $\rho_s : K_s \longrightarrow O(k_s)$ をすべて求め，(*) によって G 多様体 M' を構成して，M' が n 次元有理コホモロジー複素射影空間となるものをすべて数え挙げたいのであるが，これにはまだ日時を要する。 今のところ，定理0の (C) の場合が起り得ないことが分かっている。 更に，(A)-(i) の場合，および (B) の場合が殆んど分類できて，§3 に挙げる例に尽きそうである。 (A)-(ii) の場合が，最も難かしく，目下研究中である。

§ 1. Transformation group with codimension one orbit

1.1. Let us first recall some basic facts about differentiable transformation groups.

(1.1.1) Let G be a compact Lie group acting differentiably on a manifold M. Then by averaging an arbitrary given Riemannian metric on M, we may have a G-invariant Riemannian metric on M.

(1.1.2) Let $x \in M$. Then the isotropy subgroup G_x acts orthogonally on a normal vector space N_x of the orbit $G(x)$ at x; we call it the slice representation of G_x at x and denote by $\rho_x : G_x \longrightarrow O(N_x)$, where $O(N_x)$ is the group of orthogonal transformations on N_x.

(1.1.3) (Differentiable slice theorem) Let $E(\nu)$ be the normal bundle of the orbit $G(x) = G/G_x$. Then

$$E(\nu) = G \times_{G_x} N_x$$

where G_x acts on N_x via ρ_x. We note that G acts naturally on $E(\nu)$ as bundle mappings and we may choose small positive real number ε such that the exponential mapping gives an equivariant diffeomorphism of the ε-disk bundle of $E(\nu)$ onto an invariant tubular neighborhood of $G(x)$. (cf. [3], Lemma 3.1)

(1.1.4) Let $H \subset G$ be a closed subgroup. Denote by (H), the set of all subgroups of G which is conjugate to H in G. We introduce the following partial ordering relation " $<$ " by defining $(H_1) < (H_2)$ if and only if there exists $H_1 \in (H_1)$ and $H_2 \in (H_2)$ such that $H_1 \subset H_2$. If M is connected, then there exists an absolute minimal (H) among the conjugate classes $\{(G_x) \mid x \in M\}$, moreover the set

$$M_{(H)} = \{x \in M \mid G_x \in (H)\}$$

is a dense open submanifold. The conjugate class (H) is called the type of principal isotropy subgroups, and the orbit G/H is called principal (cf.[3],(2.2) and (2.4)). An orbit G(x) is called singular if $\dim G(x) < \dim G/H$.

Combining (1.1.3) and (1.1.4), we have the following result.

Lemma 1.1.5. *If M is connected, then the slice representation of G_x at $x \in M$ is trivial if and only if G_x is a principal isotropy subgroup.*

Corollary 1.1.6. *If M and G are connected and G(x) is an orbit of codimension one, then G(x) is a principal orbit only when the normal line bundle of G(x) in M is orientable.*

1.2. Now we prove the following result.

Lemma 1.2.1. *Let G be a compact connected Lie group. Let M be a compact connected manifold without boundary and assume*

$$H^1(M ; Z_2) = 0.$$

Suppose that G acts differentiably on M with an orbit G/K of codimension one. Then G/K is a principal orbit, and M has just two singular orbits G/K_1 and G/K_2 (equivariantly diffeomorphic or not). *Moreover there is a closed invariant tubular neighborhood X_s of G/K_s (s = 1,2) such that*

$$M = X_1 \cup X_2 \quad \underline{\text{and}} \quad X_1 \cap X_2 = \partial X_1 = \partial X_2.$$

Proof. Let N be a closed invariant tubular neighborhood of G/K in M. Consider the following commutative diagram :

$$\begin{array}{ccccc} H^0(G/K;Z_2) & \xrightarrow{\phi} & H^1(N,\partial N;Z_2) & \xleftarrow{\cong} & H^1(M,M\text{-int}N;Z_2) \\ \downarrow \cdot w_1 & & \downarrow & & \downarrow \\ H^1(G/K;Z_2) & \xleftarrow{s^*} & H^1(N\ ;\ Z_2) & \xleftarrow{i^*} & H^1(M\ ;\ Z_2). \end{array}$$

Here ϕ is a Thom isomorphism and w_1 is a first Stiefel-Whitney class of the normal line bundle of G/K in M. Then $H^1(M;Z_2) = 0$ implies $w_1 = 0$, and hence G/K is a principal orbit by Corollary 1.1.2. Next, if M has no singular orbit, then M has just one isotropy type (K), and hence there is a differentiable fibration

$$G/K \longrightarrow M \xrightarrow{p} M^*$$

where M* is the orbit space which is a circle. Then the homomorphism

$$p_* : H_1(M\ ;\ Z) \longrightarrow H_1(M^*\ ;\ Z) \cong \mathbf{Z}$$

is surjective, because G/K is connected. This fact contradicts $H^1(M\ ;\ Z_2) = 0$. Therefore M has at least one singular orbit. Then we can easily seen that M is a special G-manifold (in the sense of Hirzebruch-Mayer) with the orbit space $M^* = [1,2]$ by the differentiable slice theorem (1.1.3). Let $p : M \longrightarrow M^*$ be a natural projection. Then $p^{-1}(s)$ is a singular orbit for $s = 1,2$ and $M_{(K)} = p^{-1}((1,2))$. Moreover, let

$$X_1 = p^{-1}([1,3/2]) \quad \text{and} \quad X_2 = p^{-1}([3/2,2]).$$

Then X_s is a closed invariant tubular neighborhood of $G/K_s = p^{-1}(s)$ for $s = 1,2$. q.e.d.

§ 2. Cohomological aspect

2.1. Let M be a 2n-dimensional compact connected orientable manifold without boundary and assume

$$H^*(M ; Q) = Q[u]/(u^{n+1}), \ \deg u = 2.$$

We call such a manifold M rational cohomology complex projective n-space. Let X_1, X_2 be 2n-dimensional compact connected submanifolds of M such that

$$M = X_1 \cup X_2 \quad \text{and} \quad X_1 \cap X_2 = \partial X_1 = \partial X_2.$$

Let $f_s^* : H^*(M ; Q) \longrightarrow H^*(X_s ; Q)$ be the homomorphism induced by the inclusion map $f_s : X_s \longrightarrow M$ for $s = 1,2$. Then we have the following result.

Lemma 2.1.1. *Let* n_1, n_2 *be non-negative integers such that*

$$f_s^*(u^{n_s}) \neq 0 \quad \text{but} \quad f_s^*(u^{n_s+1}) = 0$$

for $s = 1,2$. *Then we have* $n = n_1 + n_2 + 1$.

Proof. By the following exact sequence :

$$H^{k-1}(X_s;Q) \xrightarrow{\delta_s} H^k(M,X_s;Q) \longrightarrow H^k(M;Q) \xrightarrow{f_s^*} H^k(X_s;Q)$$
$$H^k(M,X_s;Q) = H^k(X_{3-s},\partial X_{3-s};Q)$$

we have the following equations of Poincaré polynomials :

$$P(X_{3-s},\partial X_{3-s};t) = P(\ker f_s^* ;t) + P(\operatorname{im} \delta_s ;t),$$

$$P(X_s ;t) = P(\operatorname{im} f_s^* ;t) + t^{-1} P(\operatorname{im} \delta_s ;t).$$

Thus we have

(2.1.2) $P(X_{3-s},\partial X_{3-s};t) - t\, P(X_s;t) = P(\ker f_s^* ;t) - t\, P(\operatorname{im} f_s^* ;t)$

for $s = 1,2$. By the Poincaré duality for X_s , we have

$$P(X_s,\partial X_s;t) = t^{2n} P(X_s ;t^{-1}),$$

$$P(X_s ;t) = t^{2n} P(X_s,\partial X_s;t^{-1}).$$

Then we have from (2.1.2)

$$P(\ker f_1^* ;t) - t\, P(\operatorname{im} f_1^* ;t) = t^{2n}(P(\operatorname{im} f_2^* ;t^{-1}) - t\, P(\ker f_2^* ;t^{-1})).$$

By the assumption on the integers n_1, n_2 we have

$$P(\operatorname{im} f_s^* ;t) = 1 + t^2 + \dots + t^{2n_s} ,$$

$$P(\ker f_s^* ;t) = t^{2n_s+2} + \dots + t^{2n} .$$

Therefore we have

$$t^{2n_1+2} + \dots + t^{2n} - t(1 + t^2 + \dots + t^{2n_1})$$

$$= t^{2n}(1 + t^{-2} + \dots + t^{-2n_2} - t(t^{-2n_2-2} + \dots + t^{-2n})).$$

Put $t = 1$. Then we have $n = n_1 + n_2 + 1$. q.e.d.

Remark. Let $V = \bigoplus_{n \geq 0} V_n$ be a finitely generated graded module over the rational numbers Q and $b_n = \dim V_n$. Then the polynomial

$$P(V;t) = b_0 + b_1 t + b_2 t^2 + \cdots$$

is called the Poincaré polynomial of V. If $V = H^*(X;Q)$ for a topological space X, then simply denote

$$P(X;t) = P(V;t).$$

2.2. From now on, we assume that M is a simply connected rational cohomology complex projective n-space and G is a compact connected Lie group which acts differentiably on M with a codimension one orbit G/K. Then by Lemma 1.2.1, there are just two singular orbits G/K_1 and G/K_2 (we can assume $K \subset K_s$ for $s = 1,2$), moreover there is a closed invariant tubular neighborhood X_s of G/K_s $(s = 1,2)$ in M, such that

$$M = X_1 \cup X_2 \quad \text{and} \quad X_1 \cap X_2 = \partial X_1 = \partial X_2.$$

Let n_1, n_2 be non-negative integers defined in Lemma 2.1.1, and let

$$k_s = 2n - \dim G/K_s$$

for $s = 1,2$. Then it is clear that

(2.2.1) $$2 \leq k_s \leq 2(n - n_s), \ (s = 1,2).$$

Because $\partial X_s = G/K$ as G-manifolds, the fibre bundle

$$K_s/K \longrightarrow G/K \xrightarrow{p_s} G/K_s$$

is a (k_s-1)-sphere bundle.

Lemma 2.2.2. _If_ $k_2 > 2$, _then_ G/K_1 _is simply connected and hence_ K_1 _is connected_.

Proof. If $k_2 > 2$, then $\pi_1(G/K_1) = \pi_1(M)$ by the general position theorem. Thus G/K_1 is simply connected by the assumption that M is simply connected. Let K_1^0 be the identity component of K_1. Then G/K_1^0 is a connected finite covering space over a simply connected space G/K_1. Thus $K_1^0 = K_1$. q.e.d.

2.3. First we assume that G/K_1 and G/K_2 are orientable, and we have the following result.

Proposition 2.3.1. _Assume that_ G/K_1 _and_ G/K_2 _are orientable_.
(i) _If_ $k_1 \equiv k_2 \equiv 0 \pmod 2$, _then_ G/K_s _is a rational cohomology complex projective_ n_s_-space and_ $k_s = 2(n-n_s)$ _for_ $s = 1,2$.
(ii) _The case_ $k_1 \equiv k_2 \equiv 1 \pmod 2$ _does not occur_.
(iii) _If_ $k_1 \equiv 0 \pmod 2$ _and_ $k_2 \equiv 1 \pmod 2$, _then_ $k_1 + k_2 = n + 2$ _and there are two cases_ :

(a) $n_1 = n_2$ _and_

$$P(G/K_1;t) = (1 + t^{k_2-1})(1 + t^2 + \cdots + t^{2n_1}),$$

$$P(G/K_2;t) = (1 + t^{k_1-1})(1 + t^2 + \cdots + t^{2n_2}).$$

(b) $k_1 = 2(n_2 + 1)$, $k_2 = n_1 - n_2 + 1$ _and_

$$P(G/K_1\ ;t) = (1 + t^{n_1-n_2})(1 + t^2 + \dots + t^{n_1+n_2}),$$

$$P(G/K_2\ ;t) = (1 + t^n)(1 + t^2 + \dots + t^{2n_2}).$$

Proof. We have

$$P(X_s, \partial X_s\ ;t) = t^{k_s} P(G/K_s\ ;t)$$

by Thom isomorphism and

$$P(X_s\ ;t) = P(G/K_s\ ;t).$$

Thus we have from (2.1.2),

$$(2.3.2)_1 \quad P(G/K_1\ ;t) = t^{k_2-1} P(G/K_2\ ;t) + (1 + t^2 + \dots + t^{2n_1}) - t^{-1}(t^{2n_1+2} + \dots + t^{2n}),$$

$$(2.3.2)_2 \quad P(G/K_2\ ;t) = t^{k_1-1} P(G/K_1\ ;t) + (1 + t^2 + \dots + t^{2n_2}) - t^{-1}(t^{2n_2+2} + \dots + t^{2n}).$$

Because $n = n_1 + n_2 + 1$, we have from (2.3.2),

$$(2.3.3)_1 \quad (1-t^{k_1+k_2-2})P(G/K_1\ ;t) = (1-t^{k_2+2n_2})(1 + t^2 + \dots + t^{2n_1}) + (t^{k_2-1} - t^{2n_1+1})(1 + t^2 + \dots + t^{2n_2}),$$

$$(2.3.3)_2 \quad (1-t^{k_1+k_2-2})P(G/K_2\ ;t) = (1-t^{k_1+2n_1})(1 + t^2 + \dots + t^{2n_2}) + (t^{k_1-1} - t^{2n_2+1})(1 + t^2 + \dots + t^{2n_1}).$$

Put $t = -1$ in (2.3.3). We have

$$(2.3.4)\quad \begin{aligned}(1-(-1)^{k_1+k_2})\,\chi(G/K_1) &= (1-(-1)^{k_2})(n+1),\\ (1-(-1)^{k_1+k_2})\,\chi(G/K_2) &= (1-(-1)^{k_1})(n+1)\end{aligned}$$

where $\chi(G/K_s) = P(G/K_s;-1)$ is the Euler characteristic of G/K_s. In particular, $k_1 \equiv k_2 \pmod 2$ implies $k_1 \equiv k_2 \equiv 0 \pmod 2$ by (2.3.4).

(i) If $k_1 \equiv k_2 \equiv 0 \pmod 2$, then

$$\chi(G/K_s) \neq 0$$

for $s = 1,2$ from (2.3.3). Thus

$$\operatorname{rank} K_s^0 = \operatorname{rank} G$$

for $s = 1,2$ and hence

$$H^{odd}(G/K_s^0\ ;\ Q) = \bigoplus_k H^{2k+1}(G/K_s^0\ ;\ Q) = 0$$

(cf. [/],Theorem 26.1), where K_s^0 is the identity component of K_s. Because the induced homomorphism

$$H^*(G/K_s\ ;\ Q) \longrightarrow H^*(G/K_s^0\ ;\ Q)$$

is injective, the Poincaré polynomial $P(G/K_s\ ;\ t)$ is an even function for $s = 1,2$. Then we have from (2.3.2),

$$P(G/K_s\ ;\ t) = 1 + t^2 + \ldots + t^{2n_s}$$

for $s = 1,2$. Therefore G/K_s is a rational cohomology complex projective n_s-space and $k_s = 2(n - n_s)$ for $s = 1,2$.

(iii) Next, if $k_1 \equiv 0 \pmod 2$ and $k_2 \equiv 1 \pmod 2$, then

$$\chi(G/K_1) = n + 1 \neq 0 \quad \text{and} \quad \chi(G/K_2) = 0$$

from (2.3.4). Thus $P(G/K_1;t)$ is an even function, and we have from (2.3.3),

$$(2.3.5)\quad \begin{aligned} P(G/K_1;t) &= 1 + t^2 + \dots + t^{2n_1} + t^{k_2-1}(1 + t^2 + \dots + t^{2n_2}), \\ t^{k_1+k_2-2}P(G/K_1;t) &= t^{k_2+2n_2}(1 + t^2 + \dots + t^{2n_1}) \\ &\quad + t^{2n_1+1}(1 + t^2 + \dots + t^{2n_2}). \end{aligned}$$

Thus we have

$$(2.3.6)\quad \begin{aligned} &(t^{k_1+k_2-2} - t^{k_2+2n_2})(1 + t^2 + \dots + t^{2n_1}) \\ &= (t^{2n_1+1} - t^{k_1+2k_2-3})(1 + t^2 + \dots + t^{2n_2}). \end{aligned}$$

Recall that $k_1 - 2 \leqslant 2n_2$ from (2.2.1) and Lemma 2.1.1.

(iii)$_a$ First assume $k_1 - 2 < 2n_2$. Then we have

$$k_1 + k_2 - 2 = 2n_1 + 1$$

and

$$\begin{aligned} &(1 + t + \dots + t^{2n_2-k_1+1})(1 + t^2 + \dots + t^{2n_1}) \\ &= (1 + t + \dots + t^{k_2-2})(1 + t^2 + \dots + t^{2n_2}) \end{aligned}$$

from (2.3.6). Put $t = 1$. Then we have

$$\begin{aligned} (2n_2 + 2 - k_1)(n_1 + 1) &= (k_2 - 1)(n_2 + 1) \\ &= (2n_1 + 2 - k_1)(n_2 + 1), \end{aligned}$$

and hence $n_1 = n_2$. Moreover

$$k_1 + k_2 = 2n_1 + 3 = n + 2.$$

(iii)$_b$ Next assume $k_1 - 2 = 2n_2$. Then

$$2n_1 + 1 = k_1 + 2k_2 - 3$$

from (2.3.6), and hence

$$k_1 = 2(n_2 + 1) \quad \text{and} \quad k_2 = n_1 - n_2 + 1.$$

Moreover

$$k_1 + k_2 = n_1 + n_2 + 3 = n + 2.$$

The Poincaré polynomial $P(G/K_1;t)$ is obtained from (2.3.5), and $P(G/K_2;t)$ is obtained from (2.3.2) and the polynomial $P(G/K_1;t)$.

q.e.d.

2.4. Now we assume $k_1 = 2$ and consider certain relation between $H^*(G/K_s^0\,;\,Q)$ and $H^*(G/K_s\,;\,Q)$, where K_s^0 is the identity component of K_s. The following argument is essentially due to H.C.Wang [4].

Remark. If G/K_2 is non-orientable, then we have $k_1 = 2$ from (2.2.1) and Lemma 2.2.2.

Lemma 2.4.1. If $k_1 = 2$, then the induced homomorphism R_k^* is an identity on $H^*(G/K^0\,;\,Q)$ for every $k \in K$. Here the right translation R_k on G/K^0 is given by $R_k(gK^0) = gkK^0$.

Proof. (i) First assume $k_2 > 2$. Then K_1 is connected from Lemma 2.2.2 and the coset space K_1/K is a circle. Therefore there is a connected central subgroup T of K_1 such that

$$K \subset K_1 = T \cdot K^0 .$$

Hence for each $k \in K$ there is $u \in T \cap K$ such that $R_k = R_u$ on G/K^0. Because T is connected, there is a continuous mapping $u : [0,1] \longrightarrow T$ such that $u(0)$ is the identity element and $u(1) = u$. Because each $u(t)$ commutes with each element of K, a homotopy

$$H_t : G/K^0 \longrightarrow G/K^0$$

can be defined by $H_t(gK^0) = gu(t)K^0$, where H_0 is the identity and $H_1 = R_u = R_k$. Therefore R_k^* is an identity.

(ii) Next assume $k_2 = 2$. Let X_s be the invariant closed tubular neighborhood of G/K_s in M $(s = 1,2)$ such that

$$M = X_1 \cup X_2 \quad \text{and} \quad X_1 \cap X_2 = \partial X_1 = \partial X_2.$$

Let $i_s : X_1 \cap X_2 \longrightarrow X_s$ be an inclusion mapping. Then the induced homomorphism

$$i_{s*} : \pi_1(X_1 \cap X_2) \longrightarrow \pi_1(X_s)$$

is surjective for $s = 1,2$ from the general position theorem. Thus there is a natural surjection

$$h_s : \pi_1(X_s) \longrightarrow \pi_1(X_1 \cap X_2)/(\ker i_{1*})\cdot(\ker i_{2*})$$

for $s = 1,2$ such that the following diagram is commutative :

$$\begin{array}{ccc} \pi_1(X_1 \cap X_2) & \xrightarrow{i_{1*}} & \pi_1(X_1) \\ \downarrow i_{2*} & & \downarrow h_1 \\ \pi_1(X_2) & \xrightarrow{h_2} & \pi_1(X_1 \cap X_2)/(\ker i_{1*})\cdot(\ker i_{2*}) . \end{array}$$

Then there is a surjection

$$\pi_1(X_1 \cup X_2) \longrightarrow \pi_1(X_1 \cap X_2)/(\ker i_{1*})\cdot(\ker i_{2*})$$

by van Kampen's theorem. But $M = X_1 \cup X_2$ is simply connected and hence

$$\pi_1(X_1 \cap X_2) = (\ker i_{1*})\cdot(\ker i_{2*}).$$

On the other hand, the inclusion $i_s : X_1 \cap X_2 \longrightarrow X_s$ is homotopy equivalent to the projection $p_s : G/K \longrightarrow G/K_s$. Thus we have

$$\pi_1(G/K) = (\ker p_{1*})\cdot(\ker p_{2*}). \tag{2.4.2}$$

From homotopy exact sequences for the principal bundles

$$G \longrightarrow G/K \quad \text{and} \quad G \longrightarrow G/K_s$$

we have a commutative diagram :

$$\begin{array}{ccccc} \pi_1(G) & \longrightarrow & \pi_1(G/K) & \xrightarrow{\theta} & K/K^0 \\ \Big\downarrow \mathrm{id} & & \Big\downarrow p_{s*} & & \Big\downarrow \iota_s \\ \pi_1(G) & \longrightarrow & \pi_1(G/K_s) & \xrightarrow{\theta_s} & K_s/K_s^0 \end{array}$$

where θ and θ_s $(s = 1,2)$ are surjective. Thus we have from (2.4.2),

$$\begin{aligned} K/K^0 &= \theta(\pi_1(G/K)) = \theta((\ker p_{1*})\cdot(\ker p_{2*})) \\ &= \theta(\ker p_{1*})\cdot\theta(\ker p_{2*}) \subset (\ker \iota_1)\cdot(\ker \iota_2) \subset K/K^0. \end{aligned}$$

Therefore

$$(2.4.3) \qquad K/K^0 = (K_1^0 \cap K/K^0)\cdot(K_2^0 \cap K/K^0) \subset (K_1^0/K^0)\cdot(K_2^0/K^0),$$

because $\ker \zeta_s = K_s^0 \cap K/K^0$. Then the proof of Lemma 2.4.1 for $k_2 = 2$ is done similarly as for $k_2 > 2$. q.e.d.

Now we consider a commutative diagram of natural projections :

$$(2.4.4) \qquad \begin{array}{ccc} G/K^0 & \xrightarrow{\ q\ } & G/K \\ \downarrow{\scriptstyle p_s^0} & & \downarrow{\scriptstyle p_s} \\ G/K_s^0 & \xrightarrow{\ q_s\ } & G/K_s \end{array}$$

for $s = 1,2$.

Lemma 2.4.5. <u>If</u> $k_1 = 2$, <u>then</u>

$$H^*(G/K_s^0\ ;\ Q) = q_s^* H^*(G/K_s\ ;\ Q) + (\ker p_s^{0*})$$

<u>for</u> $s = 1,2$ (direct sum or not).

Proof. Because K_s/K is a $(k_s - 1)$-sphere, K_s/K is connected and hence the natural mapping $K_s^0/K^0 \longrightarrow K_s/K$ is surjective. Thus

$$(2.4.6) \qquad K_s = K_s^0\cdot K \quad (s = 1,2).$$

Hence for each $a \in K_s$ there is $k \in K$ such that $R_a^* = R_k^*$ on $H^*(G/K_s^0\ ;\ Q)$. By Lemma 2.4.1 and a commutative diagram :

$$\begin{array}{ccc} H^*(G/K_s^0\ ;\ Q) & \xrightarrow{\ p_s^{0*}\ } & H^*(G/K^0\ ;\ Q) \\ \downarrow{\scriptstyle R_a^* = R_k^*} & & \downarrow{\scriptstyle R_k^* = \mathrm{id}} \\ H^*(G/K_s^0\ ;\ Q) & \xrightarrow{\ p_s^{0*}\ } & H^*(G/K^0\ ;\ Q), \end{array}$$

we have

(2.4.7) $$p_s^{0*}(u) = p_s^{0*}(R_a^*(u))$$

for each $a \in K_s$ and each $u \in H^*(G/K_s^0 ; Q)$. By averaging (2.4.7) on a finite group K_s/K_s^0, we have

$$p_s^{0*} H^*(G/K_s^0 ; Q) = p_s^{0*} q_s^* H^*(G/K_s ; Q),$$

because

$$q_s^* H^*(G/K_s ; Q) = H^*(G/K_s^0 ; Q)^{K_s/K_s^0} .$$

Thus we have

$$H^*(G/K_s^0 ; Q) = q_s^* H^*(G/K_s ; Q) + (\ker p_s^{0*}). \quad \text{q.e.d.}$$

2.5. Denote by $J = \bigoplus_k J_k$, $J_k = q_2^* H^k(G/K_2 ; Q)$. Then J is a graded subalgebra of $H^*(G/K_2^0 ; Q)$. Because

$$K_2^0/K^0 \longrightarrow G/K^0 \xrightarrow{p_2^0} G/K_2^0$$

is an orientable $(k_2 - 1)$-sphere bundle, its rational Euler class $e(p_2^0)$ can be determined up to sign. Then

Lemma 2.5.1. $\ker p_2^{0*} = J\cdot e(p_2^0) + J\cdot e(p_2^0)^2$.

Proof. From a Gysin sequence for a sphere bundle and Lemma 2.4.5, we have

$$\ker p_2^{0*} = H^*(G/K_2^0 ; Q)\cdot e(p_2^0) = J\cdot e(p_2^0) + (\ker p_2^{0*})\cdot e(p_2^0).$$

Hence

$$\ker p_2^{0*} = J\cdot e(p_2^0) + J\cdot e(p_2^0)^2 + \ldots + J\cdot e(p_2^0)^N$$

for sufficiently large N, as submodules of $H^*(G/K_2^0\ ;\ Q)$. For each $k \in K$, we have a commutative diagram :

$$\begin{array}{ccc} G/K^0 & \xrightarrow{R_k} & G/K^0 \\ \downarrow p_2^0 & & \downarrow p_2^0 \\ G/K_2^0 & \xrightarrow{R_k} & G/K_2^0 \end{array}$$

which is a bundle mapping. Thus we have

$$R_k^* \, e(p_2^0) = e(p_2^0) \text{ or } -e(p_2^0).$$

Here $R_k^* \, e(p_2^0) = -e(p_2^0)$ occurs when R_k reverses an orientation of the sphere bundle. Therefore

$$R_k^*(e(p_2^0)^2) = e(p_2^0)^2$$

for each $k \in K$. Because

$$J = q_2^* \, H^*(G/K_2\ ;\ Q) = H^*(G/K_2^0\ ;\ Q)^{K_2} = H^*(G/K_2^0\ ;\ Q)^K$$

by (2.4.6), we have

(2.5.2) $$e(p_2^0)^2 \in J$$

and hence

$$\ker p_2^{0*} = J\cdot e(p_2^0) + J\cdot e(p_2^0)^2 . \qquad \text{q.e.d.}$$

Lemma 2.5.3. $\dim(\ker p_2^{0*}) \leqslant \dim J + \dim(J \cap \ker p_2^{0*})$.

<u>Here the equality occurs if and only if</u>

$$J\cdot e(p_2^0) \cap J\cdot e(p_2^0)^2 = 0, \quad J\cdot e(p_2^0)^2 = J \cap \ker p_2^{0*}$$

and $E : J \longrightarrow \ker p_2^{0*}$ *is injective, where* E *is defined by* $E(x) = x\cdot e(p_2^0)$.

Proof. By (2.5.2), we have

$$J\cdot e(p_2^0)^2 \subset J \cap \ker p_2^{0*}$$

and hence we have from Lemma 2.5.1

$$\dim(\ker p_2^{0*}) \leqslant \dim J + \dim(J \cap \ker p_2^{0*}).$$

Moreover we have the condition on which the equality occurs. q.e.d.

2.6. Now we assume that G/K_2 is non-orientable. Then we have $k_1 = 2$ from (2.2.1) and Lemma 2.2.2.

Lemma 2.6.1. *If* G/K_2 *is non-orientable, then*

$$P(G/K_2^0\ ;t) = (1 + t^{k_2})P(G/K_2\ ;t),$$

$$P(G/K^0\ ;t) = (1 + t^{2k_2-1})P(G/K_2\ ;t) - P(n_1,n_2\ ;t).$$

Here $P(n_1,n_2\ ;t) = 0$ *for* $n_1 \geqslant n_2$ *and*

$$P(n_1,n_2\ ;t) = t^{2n_1+1} + t^{2n_1+2} + \ldots + t^{2n_2}$$

for $n_1 < n_2$.

Proof. From a Gysin sequence :

$$H^{k-k_2}(G/K_2^0;Q) \xrightarrow{e(p_2^0)} H^k(G/K_2^0;Q) \xrightarrow{p_2^{0*}} H^k(G/K^0;Q) \xrightarrow{\delta} H^{k+1-k_2}(G/K_2^0;Q),$$

we have

$$P(G/K_2^0\ ;t) = P(\mathrm{im}\ p_2^{0*}\ ;t) + P(\ker\ p_2^{0*}\ ;t),$$

(2.6.2) $$P(G/K_2^0\ ;\ t) = t^{-k_2} P(\ker\ p_2^{0*}\ ;t) + P(\mathrm{im}\,\delta\ ;t),$$

$$P(G/K^0\ ;t) = t^{k_2-1} P(\mathrm{im}\,\delta\ ;t) + P(\mathrm{im}\ p_2^{0*}\ ;t).$$

By Lemma 2.4.5 and the definition $J = q_2^* H^*(G/K_2\ ;\ Q)$,

$$P(\mathrm{im}\ p_2^{0*}\ ;t) = P(p_2^{0*}(J)\ ;t)$$

$$= P(J\ ;t) - P(J \cap \ker\ p_2^{0*}\ ;t),$$

and hence

(2.6.3) $$P(\mathrm{im}\ p_2^{0*}\ ;t) = P(G/K_2\ ;t) - P(J \cap \ker\ p_2^{0*}\ ;t).$$

Because G/K_2 is non-orientable, there is $k \in K_2$ such that the right translation R_k on G/K_2^0 reverses an orientation of G/K_2^0. Then

(2.6.4) $$2\cdot\dim H^*(G/K_2\ ;\ Q) \leqslant \dim H^*(G/K_2^0\ ;\ Q)$$

by Poincaré duality (cf.[2]). By Lemma 2.4.5, we have

(2.6.5) $$\dim H^*(G/K_2^0\ ;\ Q) = \dim J + \dim(\ker\ p_2^{0*}) - \dim(J \cap \ker\ p_2^{0*}).$$

Then we have

$$\dim J \leqslant \dim(\ker\ p_2^{0*}) - \dim(J \cap \ker\ p_2^{0*})$$

from (2.6.4),(2.6.5) and $\dim J = \dim H^*(G/K_2\ ;\ Q)$. Thus we have

$$\dim J = \dim(\ker\ p_2^{0*}) - \dim(J \cap \ker\ p_2^{0*})$$

by Lemma 2.5.3. Moreover we have

$$\text{(2.6.6)} \qquad P(\ker p_2^{0*} ;t) = t^{k_2} P(J ;t) + P(J \cap \ker p_2^{0*} ;t)$$

from Lemma 2.5.1 and Lemma 2.5.3. Combining (2.6.2),(2.6.3) and (2.6.6), we have

$$P(G/K_2^0 ;t) = (1 + t^{k_2}) P(G/K_2 ;t),$$

$$P(G/K^0 ;t) = (1 + t^{2k_2-1}) P(G/K_2 ;t) - (1 + t^{-1}) P(J \cap \ker p_2^{0*} ;$$

It remains to show

$$(1 + t^{-1}) P(J \cap \ker p_2^{0*} ;t) = P(n_1, n_2 ;t).$$

Consider the following commutative diagram :

$$\begin{array}{ccc}
H^*(G/K_2 ; Q) & \xrightarrow{p_2^*} & H^*(G/K ; Q) \\
\downarrow{\scriptstyle q_2^*} & & \downarrow{\scriptstyle q^*} \\
H^*(G/K_2^0 ; Q) & \xrightarrow{p_2^{0*}} & H^*(G/K^0 ; Q).
\end{array}$$

Because q^* is an isomorphism from Lemma 2.4.1, we have

$$P(J \cap \ker p_2^{0*} ;t) = P(\ker p_2^* ;t).$$

Recall that $p_2 : G/K \longrightarrow G/K_2$ is homotopy equivalent to $i_2 : X_1 \cap X_2 \longrightarrow X_2$, and consider the following commutative diagram :

$$\begin{array}{ccccc}
H^*(M, X_1 ; Q) & \longrightarrow & H^*(M ; Q) & \xrightarrow{f_1^*} & H^*(X_1 ; Q) \\
\downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle f_2^*} & & \downarrow{\scriptstyle i_1^*} \\
H^*(X_2, X_1 \cap X_2 ;Q) & \longrightarrow & H^*(X_2 ;Q) & \xrightarrow{i_2^*} & H^*(X_1 \cap X_2 ;Q).
\end{array}$$

Then we have

$$P(\ker p_2^{0*}\,;t) = P(\ker i_2^*\,;t) = \begin{cases} t^{2n_1+2} + \dots + t^{2n_2} & (\text{if } n_1 < n_2) \\ 0 \quad (\text{if } n_1 \geqslant n_2). \end{cases}$$

Thus we have

$$(1 + t^{-1})P(J \cap \ker p_2^{0*}\,;t) = \begin{cases} t^{2n_1+1} + \dots + t^{2n_2} & (\text{if } n_1 < n_2) \\ 0 \quad (\text{if } n_1 \geqslant n_2). \end{cases}$$

q.e.d.

2.7. Now we can prove the following result.

Proposition 2.7.1. *Assume that* n is even, G/K_2 *is non-orientable.*
(i) *If* G/K_1 *is orientable, then* G/K_1 *is a rational cohomology complex projective* $(n - 1)$*-space and* G/K_2^0 *is a rational cohomology* n*-sphere.*
(ii) *If* G/K_1 *is non-orientable, then* $n = 3$ *and*

$$P(G/K_s\,;t) = 1 + t^2, \quad P(G/K_s^0\,;t) = (1 + t^2)^2$$

for $s = 1,2$.

Proof. Because G/K_2 is non-orientable, we have

$$k_1 = 2 \quad \text{and} \quad \dim G/K_1 = 2n - 2.$$

(i) First assume that G/K_1 is orientable. Then by the Poincaré duality for G/K_1, we have from (2.3.2),

$$(2.7.2) \qquad t^{2n-1} P(G/K_2\,;t^{-1}) = P(G/K_1\,;t) + t^{2n_1+1}(1 + t^2 + \dots + t^{2n_2}) - (1 + t^2 + \dots + t^{2n_1}).$$

By the Poincaré duality for G/K_2^0, we have from Lemma 2.6.1,

(2.7.3) $$t^{2n} P(G/K_2 ; t^{-1}) = t^{2k_2} P(G/K_2 ; t).$$

Combining (2.7.2),(2.7.3) and (2.3.2) with $k_1 = 2$, we have

(2.7.4) $$(1 - t^{2k_2})P(G/K_1 ; t) = (1 - t^{2k_2+2n_2})(1 + t^2 + \dots + t^{2n_1}) + (t^{2k_2-1} - t^{2n_1+1})(1 + t^2 + \dots + t^{2n_2}).$$

In particular we have

$$\chi(G/K_1) = P(G/K_1 ; -1) \neq 0.$$

Therefore $P(G/K_1 ; t)$ is an even function by the same argument as in the proof of Proposition 2.3.1 (i). Hence we have from (2.7.4),

(2.7.5) $$k_2 = n_1 + 1 \quad \text{and} \quad P(G/K_1 ; t) = 1 + t^2 + \dots + t^{2n-2}.$$

Then we have from (2.3.2) and (2.7.5),

(2.7.6) $$P(G/K_2 ; t) = 1 + t + t^2 + \dots + t^{2n_2}.$$

Thus $\chi(G/K_2) \neq 0$ and hence $P(G/K_2 ; t)$ is an even function. Therefore

$$n_2 = 0 \quad \text{and} \quad P(G/K_2 ; t) = 1$$

from (2.7.6), and hence $n_1 = n - 1$ by Lemma 2.1.1. Then

$$P(G/K_2^0 ; t) = 1 + t^n$$

from Lemma 2.6.1. Consequently G/K_1 is a rational cohomology complex projective $(n - 1)$-space and G/K_2^0 is a rational cohomology n-sphere. Moreover $\chi(G/K_2) \neq 0$ implies $n \equiv 0 \pmod 2$.

(ii) Next assume that G/K_1 is non-orientable. Then

$$k_1 = k_2 = 2.$$

From Lemma 2.6.1, we have

$$(2.7.7)\quad \begin{aligned} P(G/K_2^0\ ;t) &= (1 + t^2)P(G/K_2\ ;t), \\ P(G/K^0\ ;t) &= (1 + t^3)P(G/K_2\ ;t) - P(n_1,n_2\ ;t). \end{aligned}$$

Similarly we have

$$(2.7.8)\quad \begin{aligned} P(G/K_1^0\ ;t) &= (1 + t^2)P(G/K_1\ ;t), \\ P(G/K^0\ ;t) &= (1 + t^3)P(G/K_1\ ;t) - P(n_2,n_1\ ;t). \end{aligned}$$

Here $P(a,b\ ;t) = 0$ for $a \geqslant b$ and

$$P(a,b\ ;t) = t^{2a+1} + t^{2a+2} + \ldots + t^{2b}$$

for $a < b$.

If $n_1 < n_2$, then we have from (2.7.7) and (2.7.8),

$$t^{2n_1+1} + t^{2n_1+2} + \ldots + t^{2n_2} \equiv 0 \pmod{1 + t^3}.$$

Thus $n_1 \equiv n_2 \pmod 3$ and

$$(2.7.9)\quad P(G/K_2\ ;t)-P(G/K_1\ ;t) = t^{2n_1+1}(1+t+t^2)(1+t^6+t^{12}+\ldots+t^{2(n_2-n_1-3)})$$

Then

$$(2.7.10)\qquad \chi(G/K_2) - \chi(G/K_1) = (n_1 - n_2)/3 < 0.$$

If $\chi(G/K_s) \neq 0$ for $s = 1,2$ then $P(G/K_s\ ;t)$ is an even function for $s = 1,2$ and this contradicts (2.7.9). Thus

$$\chi(G/K_1) \neq 0 \quad \text{and} \quad \chi(G/K_2) = 0$$

from (2.7.10), and hence

$$\text{(2.7.11)} \qquad \text{rank } K_1^0 = \text{rank } G \neq \text{rank } K_2^0 .$$

On the other hand,

$$\text{rank } K_s^0 = \text{rank } K^0 + 1$$

for $s = 1,2$ because $K_s/K = S^1$. This contradicts (2.7.11). Therefore the case $n_1 < n_2$ does not occur. Similarly the case $n_2 < n_1$ does not occur.

Finally if $n_1 = n_2$, then $n = 2n_1 + 1$ and we have from (2.7.7) and (2.7.8)

$$\text{(2.7.12)} \qquad \begin{aligned} P(G/K_s^0 ;t) &= (1 + t^2)P(G/K_s ;t), \\ P(G/K^0 ;t) &= (1 + t^3)P(G/K_s ;t) \end{aligned}$$

for $s = 1,2$. Let X_s be the invariant closed tubular neighborhood of G/K_s such that

$$M = X_1 \cup X_2 \quad \text{and} \quad X_1 \cap X_2 = \partial X_1 = \partial X_2.$$

Consider the Mayer-Vietoris cohomology sequence of a triad $(M ; X_1, X_2)$. Then we have

$$P(G/K_1 ;t) + P(G/K_2 ;t) - P(G/K ;t) = (1-t^{2n_1+1})(1 + t^2 + \dots + t^{2n_1}$$

Because $P(G/K ;t) = P(G/K^0 ;t)$ from Lemma 2.4.1, we have from (2.7.12),

$$\text{(2.7.13)} \qquad (1 - t^3)P(G/K_1 ;t) = (1 - t^{2n_1+1})(1 + t^2 + \dots + t^{2n_1}).$$

Thus $\chi(G/K_1) = n_1 + 1 \neq 0$ and hence $P(G/K_1\ ;t)$ is an even function. Therefore we have from (2.7.13),

$$n_1 = 1 \quad \text{and} \quad P(G/K_1\ ;t) = 1 + t^2.$$

Consequently, $n = 3$ and

$$P(G/K_s\ ;t) = 1 + t^2,$$

$$P(G/K_s^0\ ;t) = (1 + t^2)^2$$

for $s = 1,2$ from (2.7.12). q.e.d.

§3. Examples

(例1) $n = n_1 + n_2 + 1$ として，$P_n(\mathbb{C}) = P(\mathbb{C}^{n_1+1} \oplus \mathbb{C}^{n_2+1})$ 上の $U(n_1+1) \times U(n_2+1)$ の自然な作用を考える。 この作用は，余次元1の多様体

$$X = \{(u_0 : \cdots : u_{n_1} : v_0 : \cdots : v_{n_2}) \mid |u_0|^2 + \cdots + |u_{n_1}|^2 = |v_0|^2 + \cdots + |v_{n_2}|^2\}$$

に推移的に働き，特異軌道

$$P_{n_1}(\mathbb{C}) = \{(u_0 : \cdots : u_{n_1} : 0 : \cdots : 0)\},$$

$$P_{n_2}(\mathbb{C}) = \{(0 : \cdots : 0 : v_0 : \cdots : v_{n_2})\}$$

を持つ。 $U(n_1+1) \times U(n_2+1)$ の部分群 G で，X に推移的に作用するものについても同様である。 これは，定理0の (A)-(i) の場合である。

(例2) $P_n(\mathbb{C}) = P(\mathbb{R}^{n+1} \otimes_{\mathbb{R}} \mathbb{C})$ 上の $SO(n+1)$ の自然な作用を考える。 点 $(0 : \cdots : 0 : t : \sqrt{-1})$ におけるイソトロピー群を H_t とおけば，

$$H_0 = S(O(n) \times O(1)), \quad H_1 = SO(n-1) \times SO(2),$$

$$H_t = SO(n-1) \times \mathbb{Z}_2, \quad (0 < t < 1)$$

となり，余次元1の軌道を持ち，$SO(n+1)/H_0 = P_n(\mathbb{R})$ である。従って，$n \equiv 1 \pmod 2$ のとき，定理0の (A)-(ii)-(b) の場合で，$n \equiv 0 \pmod 2$ のとき，(B)の場合である。 どちらの場合も，$k_1 = 2$，$k_2 = n = n_1 + 1$，$n_2 = 0$ となっている。

(例3)　$n=2p+1$ として，$P_n(\mathbb{C})=P(\mathbb{C}^{p+1}\underset{\mathbb{C}}{\otimes}\mathbb{C}^2)$ 上の $SU(p+1)\times SU(2)$ のテンソル積による自然な作用を考える。$t\cdot e_p\otimes e_1+e_{p+1}\otimes e_2$ が表わす点のイソトロピー群を H_t とおけば，

$$H_0=S(U(p)\times U(1))\times S(U(1)\times U(1)),$$

$$H_1=\left\{\begin{pmatrix}* & 0\\ 0 & \lambda\bar{A}\end{pmatrix}\times A;|\lambda|=1\right\},\ \underset{0<t<1}{H_t}=\left\{\begin{pmatrix}* & 0\\ 0 & \lambda\bar{A}\end{pmatrix}\times A;|\lambda|=1,\ A\in S(U(1)\times U(1))\right\}$$

となり，余次元1の軌道を持つ。　これは，定理0の(A)-(ii)-(a)の場合であり，$k_1=2p,\ k_2=3,\ n=2p+1,\ n_1=n_2=p$ となっている。

(例4)　$Q_n=SO(n+2)/SO(n)\times SO(2)$ 上に，左移動によって $SO(n+1)$ を作用させる。

$$A_\theta=\begin{pmatrix}I_{n-1} & & & \\ & \cos\theta & 0 & \sin\theta\\ & 0 & 1 & 0\\ & -\sin\theta & 0 & \cos\theta\end{pmatrix}\in SO(n+2)$$

が表わす剰余類におけるイソトロピー群を H_θ とおけば，

$$H_0=SO(n),\ H_{\frac{\pi}{2}}=SO(n-1)\times SO(2),$$

$$H_\theta=SO(n-1),\ 0<\theta<\frac{\pi}{2}$$

となり，余次元1の軌道を持つ。　n が奇数で，$n\neq 1$ のとき，Q_n は単連結な n 次元有理コホモロジー複素射影空間であり，$\pi_n(Q_n)=Z_2$ である。　これは，定理0の(A)-(ii)-(b)の場

合であり，$k_1=2$，$k_2=n=n_1+1$，$n_2=0$ となっている。

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