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正準相関係数の分布の漸近展開

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§1. 単根の場合　$p(=p_1+p_2)$ 次元正規分布において始めの $p_1(\le p_2)$ 成分に関する正準相関係数を $1>\rho_1>\cdots>\rho_{p_1}>0$ とし大きさ $n+1$ の任意標本から作られる標本正準相関係数を $r_1>\cdots>r_{p_1}>0$ とおく．さらに $P^2=\mathrm{diag}(\rho_1^2,\cdots,\rho_{p_1}^2)$ $\Gamma=I-P^2$, $\Theta=\Gamma^{-1}-I$ とおく．標準形で考えれば $r_\alpha^2=ch_\alpha S_h(S_e+S_h)^{-1}$ と表わされ，S_e は Wishart 分布 $W_{p_1}(I,n-p_2)$ に従い，S_h は非心 Wishart 分布 $W_{p_1}(I\ p_2,\Omega)$ に従う．Ω が与えられたとき S_e と S_h は独立，Ω の分布は $W_{p_1}(\Theta,n)$ で与えられる．筆者および長尾〔3〕による次の結果は独立性の検定（正準相関係数の関数で与えられるもの）に対する漸近展開を行うためのものであったが，同時分布の漸近展開を求める今の場合にも役に立つ．ただし重根の場合も扱うため，A, B が対角行列と仮定していたところを一般の対称行列 A, B で成り立つよう修正する必要が生じた。

補助定理 正値対称行列 $\Lambda_e=(\lambda_{ij}^{(e)})$, $\Lambda_h=(\lambda_{ij}^{(h)})$ に関して Taylor 展開可能な関数 $f(\Lambda_e,\Lambda_h)$ について

(1) $\partial_e=\left(\frac{1}{2}(1+\delta_{ij})\frac{\partial}{\partial\lambda_{ij}^{(e)}}\right)$, $\partial_h=\left(\frac{1}{2}(1+\delta_{ij})\frac{\partial}{\partial\lambda_{ij}^{(h)}}\right)$

とおけば, 任意の対称行列 A, B $(p_1\times p_1)$ について

(2) $$E\left[\mathrm{etr}\,it\left\{A\sqrt{m}\left(\frac{S_e}{m}-I\right)+B\sqrt{m}\left(\frac{S_h}{m}-\Theta\right)\right\}\cdot f\left(\frac{S_e}{m},\frac{S_h}{m}\right)\right]=\mathrm{etr}-t^2\{A^2+(\Gamma^{-1}B)^2$$
$$-B^2\}\left\{1+\frac{1}{\sqrt{m}}\sum_1^2 d_{2\alpha-1}(it)^{2\alpha-1}+\frac{1}{m}\sum_0^3 g_{2\alpha}(it)^{2\alpha}\right\}f(\Lambda_e,\Lambda_h)\Big|_{\Lambda_e=I,\Lambda_h=\Theta}+O\left(\frac{1}{m^{3/2}}\right)$$

ただし $m=n-2\Delta$ (Δ は補正項)

$$d_1=2\,\mathrm{tr}\left\{A\partial_e+\Gamma^{-1}B\Gamma^{-1}\partial_h-B\partial_h+\left(\Delta-\frac{p_2}{2}\right)(A-B)+\Delta\Gamma^{-1}B\right\}$$

$$d_3=\frac{4}{3}\,\mathrm{tr}\left\{A^3+(\Gamma^{-1}B)^3-B^3\right\}$$

$$g_0=\mathrm{tr}\left\{\partial_e^2-\partial_h^2+(\Gamma^{-1}\partial_h)^2+(2\Delta-p_2)(\partial_e-\partial_h)+2\Delta\Gamma^{-1}\partial_h\right\}$$

(3) $$g_2=\mathrm{tr}\left\{4A^2\partial_e+4(\Gamma^{-1}B)^2\Gamma^{-1}\partial_h-4B^2\partial_h+(2\Delta-p_2)(A^2-B^2)+2\Delta(\Gamma^{-1}B)^2+\frac{1}{2}d_1^2\right.$$

$$g_4=2\,\mathrm{tr}\left\{A^4+(\Gamma^{-1}B)^4-B^4\right\}+d_1d_3$$

$$g_6=\frac{1}{2}d_3^2$$

これにより $\frac{1}{m}S_e$ は I に $\frac{1}{m}S_h$ は Θ に確率収束することがわかるから筆者〔4〕,〔5〕でやった如く摂動法により $\frac{1}{m}S_e=I$, $\frac{S_h}{m}=\Theta$ の周りで γ_α^2 の Taylor 展開を求めれば, $\frac{1}{m}S_e-I=(b_{ij})$, $\frac{1}{m}S_h-\Theta=(a_{ij})$ とおいて

(4) $$\gamma_\alpha^2=\rho_\alpha^2+\frac{a_{\alpha\alpha}-\theta_\alpha b_{\alpha\alpha}}{(1+\theta_\alpha)^2}-\frac{(a_{\alpha\alpha}-\theta_\alpha b_{\alpha\alpha})(a_{\alpha\alpha}+b_{\alpha\alpha})}{(1+\theta_\alpha)^3}+\frac{1}{(1+\theta_\alpha)^2}\sum_{j\neq\alpha}\frac{(a_{\alpha j}-\theta_\alpha b_{\alpha j})^2}{\theta_\alpha+\theta_j}$$
$$+\frac{(a_{\alpha\alpha}-\theta_\alpha b_{\alpha\alpha})(a_{\alpha\alpha}+b_{\alpha\alpha})^2}{(1+\theta_\alpha)^4}-\frac{a_{\alpha\alpha}+b_{\alpha\alpha}}{(1+\theta_\alpha)^3}\sum_{j\neq\alpha}\frac{(a_{\alpha j}-\theta_\alpha b_{\alpha j})^2}{\theta_\alpha-\theta_j}-\frac{(a_{\alpha\alpha}-\theta_\alpha b_{\alpha\alpha})}{(1+\theta_\alpha)^3}$$
$$\sum_{j\neq\alpha}\frac{(a_{\alpha j}+b_{\alpha j})(a_{j\alpha}-\theta_\alpha b_{j\alpha})}{\theta_\alpha-\theta_j}-\frac{a_{\alpha\alpha}-\theta_\alpha b_{\alpha\alpha}}{(1+\theta_\alpha)^2}\sum_{j\neq\alpha}\frac{(a_{\alpha j}-\theta_\alpha b_{\alpha j})(a_{j\alpha}-\theta_j b_{j\alpha})}{(\theta_\alpha-\theta_j)^2}$$
$$+\frac{1}{(1+\theta_\alpha)^2}\sum_{\substack{j\neq\alpha\\k\neq\alpha}}\frac{(a_{\alpha j}-\theta_\alpha b_{\alpha j})(a_{jk}-\theta_\alpha b_{jk})(a_{k\alpha}-\theta_\alpha b_{k\alpha})}{(\theta_\alpha-\theta_j)(\theta_\alpha-\theta_k)}+(4\text{次の項})$$

これより正準相関係数を2乗したもの $r_1^2, \cdots, r_{p_1}^2$ の同時分布の特性関数が次のように展開される．

$$(5)\quad E\left[\exp\left\{i\sqrt{m}\sum_{1}^{p_1} t_\alpha (r_\alpha^2-\rho_\alpha^2)\right\}\right] = E\left[\exp\left\{i\sqrt{m}\sum_{1}^{p_1} t_\alpha \frac{(a_{\alpha\alpha}-\vartheta_\alpha b_{\alpha\alpha})}{(1+\vartheta_\alpha)^2}\right\}\right.$$

$$\left.\cdot\left\{1+\frac{K_1}{\sqrt{m}}+\frac{1}{m}\left(K_2+\frac{K_1^2}{2}\right)\right\}+O\left(\frac{1}{m\sqrt{m}}\right)\right]$$

$$K_1 = im\sum_\alpha \frac{t_\alpha}{(1+\vartheta_\alpha)^2}\left\{-\frac{(a_{\alpha\alpha}-\vartheta_\alpha b_{\alpha\alpha})(a_{\alpha\alpha}+b_{\alpha\alpha})}{(1+\vartheta_\alpha)}+\sum_{j\neq\alpha}\frac{(a_{\alpha j}-\vartheta_\alpha b_{\alpha j})^2}{\vartheta_\alpha-\vartheta_j}\right\}$$

$$(6)\quad K_2 = im\sqrt{m}\sum_\alpha \frac{t_\alpha}{(1+\vartheta_\alpha)^2}\left\{\frac{(a_{\alpha\alpha}-\vartheta_\alpha b_{\alpha\alpha})(a_{\alpha\alpha}+b_{\alpha\alpha})^2}{(1+\vartheta_\alpha)^2}-\frac{a_{\alpha\alpha}+b_{\alpha\alpha}}{1+\vartheta_\alpha}\sum_{j\neq\alpha}\frac{(a_{\alpha j}-\vartheta_\alpha b_{\alpha j})^2}{\vartheta_\alpha-\vartheta_j}\right.$$

$$-\frac{a_{\alpha\alpha}-\vartheta_\alpha b_{\alpha\alpha}}{1+\vartheta_\alpha}\sum_{j\neq\alpha}\frac{(a_{\alpha j}+b_{\alpha j})(a_{\alpha j}-\vartheta_\alpha b_{\alpha j})}{\vartheta_\alpha-\vartheta_j}-(a_{\alpha\alpha}-\vartheta_\alpha b_{\alpha\alpha})\sum_{j\neq\alpha}\frac{(a_{\alpha j}-\vartheta_\alpha b_{\alpha j})(a_{j\alpha}-\vartheta_j b_{j\alpha})}{(\vartheta_\alpha-\vartheta_j)^2}$$

$$\left.+\sum_{\substack{j\neq\alpha\\k\neq\alpha}}\frac{(a_{j\alpha}-\vartheta_\alpha b_{j\alpha})(a_{jk}-\vartheta_\alpha b_{jk})(a_{k\alpha}-\vartheta_\alpha b_{k\alpha})}{(\vartheta_\alpha-\vartheta_j)(\vartheta_j-\vartheta_k)}\right\}$$

各項の平均は補助定理を使って計算できる．簡単のため $\xi = i\sqrt{m}\sum_1^{p_1} t_\alpha (a_{\alpha\alpha}-\vartheta_\alpha b_{\alpha\alpha})/(1+\vartheta_\alpha)^2$， $\varphi(t)=\exp\{-2\sum_1^{p_1}\rho_\alpha^2(1-\rho_\alpha^2)^2 t_\alpha^2\}$ とおけば

$$(7)\quad E[\exp\xi] = \varphi(t)\left[1+\frac{1}{\sqrt{m}}(i d_1+i^3 d_3)+\frac{1}{m}\left\{i^2\left(\frac{1}{2}d_1^2+\sum_\alpha\{(4\Delta-2p_2)\rho_\alpha^2+p_2\}\right.\right.\right.$$

$$\left.\cdot(1-\rho_\alpha^2)^2 t_\alpha^2\right)$$

$$\left.\left.+i^4\left(d_1 d_3+4\sum_\alpha \rho_\alpha^2(1-\rho_\alpha^2)^4(2-3\rho_\alpha^2+2\rho_\alpha^4)t_\alpha^4\right)+i^6\frac{d_3^2}{2}\right\}+O\left(\frac{1}{m\sqrt{m}}\right)\right]$$

ただし $d_1 = p_2\sum_1^{p_1}(1-\rho_\alpha^2)t_\alpha$， $d_3 = 4\sum_1^{p_1}\rho_\alpha^2(1-\rho_\alpha^2)^4 t_\alpha^3$ とする．

$$E[K_1\exp\xi] = \varphi(t)\left[i\left\{-2\sum_\alpha\rho_\alpha^2(1-\rho_\alpha^2)t_\alpha+\sum_{\alpha\neq j}\frac{1-\rho_\alpha^2}{\rho_\alpha^2-\rho_j^2}(\rho_\alpha^2+\rho_j^2-2\rho_\alpha^2\rho_j^2)t_\alpha\right\}\right.$$

$$\left.-8i^3\sum_\alpha\rho_\alpha^4(1-\rho_\alpha^2)^3 t_\alpha^3\right]+\frac{1}{\sqrt{m}}\varphi(t)\left[i^2\left\{-8\sum_\alpha\frac{\vartheta_\alpha(\vartheta_\alpha+3)}{(1+\vartheta_\alpha)^4}t_\alpha^2\right.\right.$$

$$+2\sum_{\alpha\neq j}\frac{t_\alpha}{(1+\vartheta_\alpha)^2(\vartheta_\alpha-\vartheta_j)}\left\{-\frac{1+\vartheta_\alpha^3}{(1+\vartheta_\alpha)^2}t_\alpha-\frac{1+\vartheta_j\vartheta_\alpha^2}{(1+\vartheta_j)^2}t_j+t_\alpha(1+\vartheta_j)+t_j(1+\vartheta_\alpha)\right\}$$

$$(8)\quad +p_2\left(\sum_\alpha\frac{t_\alpha}{1+\vartheta_\alpha}\right)\left\{-2\sum_\alpha\rho_\alpha^2(1-\rho_\alpha^2)t_\alpha+\sum_{\alpha\neq j}\frac{1-\rho_\alpha^2}{\rho_\alpha^2-\rho_j^2}(\rho_\alpha^2+\rho_j^2-2\rho_\alpha^2\rho_j^2)t_\alpha\right\}$$

$$\left.-2(4\Delta+p_2)\sum_\alpha\frac{\vartheta_\alpha t_\alpha^2}{(1+\vartheta_\alpha)^3}\right\}+i^4\left\{-8\sum_\alpha\frac{\vartheta_\alpha^2(2\vartheta_\alpha+9)}{(1+\vartheta_\alpha)^7}t_\alpha^4\right.$$

$$+4\sum_\alpha\frac{\vartheta_\alpha t_\alpha^3}{(1+\vartheta_\alpha)^5}\left\{-2\sum_\alpha\rho_\alpha^2(1-\rho_\alpha^2)t_\alpha+\sum_{j\neq\alpha}\frac{1-\rho_\alpha^2}{\rho_\alpha^2-\rho_j^2}(\rho_\alpha^2+\rho_j^2-2\rho_\alpha^2\rho_j^2)t_\alpha\right\}$$

$$-8P_2\sum_\alpha(1-\rho_\alpha^2)t_\alpha\cdot\sum_\alpha\rho_\alpha^4(1-\rho_\alpha^2)^3t_\alpha^3\Big\}-32i^6\sum_\alpha\rho_\alpha^2(1-\rho_\alpha^2)^4t_\alpha^3\cdot\sum\rho_\alpha^4(1-\rho_\alpha^2)^3t_\alpha^3\Big]+O(m^{-1})$$

$$E[K_2\exp S]=\varphi(t)\Big[i^2\Big\{8\sum_\alpha\frac{\theta_\alpha(2\theta_\alpha+1)}{(1+\theta_\alpha)^4}t_\alpha^2-2\sum_{j\neq\alpha}\frac{\theta_\alpha(\theta_\alpha+3\theta_j)t_\alpha^2}{(1+\theta_\alpha)^3(\theta_\alpha-\theta_j)}$$

(9) $$-4\sum_{j\neq\alpha}\frac{\theta_\alpha(\theta_\alpha+\theta_j+2\theta_\alpha\theta_j)}{(1+\theta_\alpha)^3(\theta_\alpha-\theta_j)^2}t_\alpha^2+2\sum_{j\neq\alpha}\frac{\theta_j(\theta_\alpha+\theta_j)(\theta_\alpha+\theta_j+2)t_\alpha t_j}{(1+\theta_\alpha)(1+\theta_j)^2(\theta_\alpha-\theta_j)^2}\Big\}$$

$$+i^4\cdot16\sum_\alpha\frac{\theta_\alpha^3t_\alpha^4}{(1+\theta_\alpha)^7}\Big]$$

$$E\Big[\frac{K_1^2}{2}\exp S\Big]=\varphi(t)\Big[i^2\Big\{2\sum\frac{\theta_\alpha(3\theta_\alpha+2)t_\alpha^2}{(1+\theta_\alpha)^4}+2\Big\{\sum_\alpha\frac{\theta_\alpha t_\alpha}{1+\theta_\alpha}-\frac12\sum_{\alpha\neq j}\frac{(\theta_\alpha+\theta_j)t_\alpha}{(1+\theta_\alpha)(\theta_\alpha-\theta_j)}\Big\}^2$$

$$+\frac12\sum_{\alpha\neq\beta}\frac{1}{(\theta_\alpha-\theta_\beta)^2}\Big\{\frac{\theta_\alpha^2t_\alpha}{(1+\theta_\alpha)^2}-\frac{\theta_\beta^2t_\beta}{(1+\theta_\beta)^2}\Big\}^2+\sum_{\alpha\neq\beta}\frac{\theta_\alpha+\theta_\beta+\theta_\alpha\theta_\beta}{(\theta_\alpha-\theta_\beta)^2}\Big\{\frac{\theta_\alpha t_\alpha}{(1+\theta_\alpha)^2}-\frac{\theta_\beta t_\beta}{(1+\theta_\beta)^2}\Big\}^2$$

(10) $$+\frac12\sum_{\alpha\neq\beta}\frac{(\theta_\alpha+\theta_\beta+\theta_\alpha\theta_\beta)^2}{(\theta_\alpha-\theta_\beta)^2}\Big\{\frac{t_\alpha}{(1+\theta_\alpha)^2}-\frac{t_\beta}{(1+\theta_\beta)^2}\Big\}^2\Big\}$$

$$+i^4\Big\{8\sum_\alpha\frac{\theta_\alpha^2(5\theta_\alpha+2)}{(1+\theta_\alpha)^7}t_\alpha^4+16\sum_\alpha\frac{\theta_\alpha t_\alpha}{(1+\theta_\alpha)^2}\cdot\sum_\alpha\frac{\theta_\alpha^2t_\alpha^3}{(1+\theta_\alpha)^5}$$

$$-8\sum_\alpha\frac{\theta_\alpha^2t_\alpha^3}{(1+\theta_\alpha)^5}\cdot\sum_{\alpha\neq j}\frac{(\theta_\alpha+\theta_j)t_\alpha}{(1+\theta_\alpha)(\theta_\alpha-\theta_j)}\Big\}+i^6\,32\Big(\sum_\alpha\frac{\theta_\alpha^2t_\alpha^3}{(1+\theta_\alpha)^5}\Big)^2\Big]$$

（７）～（10）を加えて $(r_1^2-\rho_1^2,\cdots,r_{p_1}^2-\rho_{p_1}^2)\sqrt{m}$ の特性関数が次のように評価される．

$$\exp\Big\{-2\sum_\alpha\rho_\alpha^2(1-\rho_\alpha^2)^2t_\alpha^2\Big\}\cdot\Big\{1+\frac{1}{\sqrt m}(id_1+i^3d_3)+\frac1m(i^2g_2+i^4g_4+i^6g_6)+O(m^{-3/2})\Big\}$$

$$d_1=P_2\sum_\alpha t_\alpha(1-\rho_\alpha^2)-2\sum t_\alpha(1-\rho_\alpha^2)\rho_\alpha^2+\sum_{j\neq\alpha}\frac{1-\rho_\alpha^2}{\rho_\alpha^2-\rho_j^2}(\rho_\alpha^2+\rho_j^2-2\rho_\alpha^2\rho_j^2)t_\alpha$$

$$d_3=4\sum_\alpha(1-\rho_\alpha^2)^3\rho_\alpha^2(1-3\rho_\alpha^2)t_\alpha^3$$

(11) $$g_2=\frac12d_1^2+\sum_\alpha\{-4(\Delta+P_2)\rho_\alpha^2+P_2\}(1-\rho_\alpha^2)^2t_\alpha^2+2\sum_\alpha\rho_\alpha^2(1-\rho_\alpha^2)(13\rho_\alpha^2-6)t_\alpha^2$$

$$+\sum_{\alpha\neq\beta}\frac{(\theta_\alpha^2+\theta_\beta^2-6\theta_\alpha\theta_\beta)t_\alpha^2}{(1+\theta_\alpha)^2(\theta_\alpha-\theta_\beta)^2}-\sum_{\alpha\neq\beta}\frac{(\theta_\alpha+\theta_\beta+2\theta_\alpha\theta_\beta)(\theta_\alpha^2+\theta_\beta^2+\theta_\alpha\theta_\beta)}{(1+\theta_\alpha)^2(1+\theta_\beta)^2(\theta_\alpha-\theta_\beta)^2}t_\alpha t_\beta$$

$$+\sum_{\alpha\neq\beta}\frac{\{-4\theta_\alpha(\theta_\alpha+\theta_\beta)+2\theta_\beta\}}{(1+\theta_\alpha)^3(\theta_\alpha-\theta_\beta)}t_\alpha^2-\sum_{\alpha\neq\beta}\frac{\theta_\alpha\theta_\beta t_\alpha t_\beta}{(1+\theta_\alpha)^2(1+\theta_\beta)^2}-\sum_{\alpha\neq\beta}\frac{t_\alpha t_\beta}{(1+\theta_\alpha)(1+\theta_\beta)}$$

$$g_4=d_1d_3+4\sum_\alpha\rho_\alpha^2(1-\rho_\alpha^2)^4(1-2\rho_\alpha^2)(2-13\rho_\alpha^2)t_\alpha^4$$

$$g_6=\frac12d_3^2$$

これを反転して次の漸近展開を得る．

定理１．$\Phi(x)$ を標準正規分布の分布関数 $\psi_j(x_\alpha)=\dfrac{\Phi^{(j)}(x_\alpha)}{c_\alpha^j\,\Phi(x_\alpha)}$,

$\tau_\alpha^2 = 4\rho_\alpha^2(1-\rho_\alpha^2)^2$ とおく．　$1>\rho_1>\cdots>\rho_{p_1}>0$ のとき

$$P\Big(\bigcap_{\alpha=1}^{p_1} \sqrt{m}(r_\alpha^2-\rho_\alpha^2)/\tau_\alpha < x_\alpha\Big) = \Big\{\prod_{\alpha=1}^{p_1}\Phi(x_\alpha)\Big\}\Big[1-\frac{1}{\sqrt{m}}(d_1+d_3)+\frac{1}{m}(g_2+g_4+g_6) + 0\Big(\frac{1}{m\sqrt{m}}\Big)\Big]$$

$$d_1 = P_2\sum_\alpha(1-\rho_\alpha^2)\psi_1(x_\alpha) - 2\sum_\alpha \rho_\alpha^2(1-\rho_\alpha^2)\psi_1(x_\alpha) + \sum_{j\neq\alpha}\frac{1-\rho_\alpha^2}{\rho_\alpha^2-\rho_j^2}(\rho_\alpha^2+\rho_j^2-2\rho_\alpha^2\rho_j^2)\psi_1(x_\alpha)$$

$$d_3 = 4\sum_\alpha(1-\rho_\alpha^2)^3\rho_\alpha^2(1-3\rho_\alpha^2)\psi_3(x_\alpha)$$

$$g_2 = \tilde{g}_2 + \frac{1}{2}\sum_\alpha(P_2-2\rho_\alpha^2)^2(1-\rho_\alpha^2)^2(\psi_2(x_\alpha)-\psi_1(x_\alpha)^2)$$

(12) $$+\sum_{\alpha\neq j}\frac{(P_2-2\rho_\alpha^2)(1-\rho_\alpha^2)^2(\rho_\alpha^2+\rho_j^2-2\rho_\alpha^2\rho_j^2)}{\rho_\alpha^2-\rho_j^2}(\psi_2(x_\alpha)-\psi_1(x_\alpha)^2)$$

$$+\frac{1}{2}\sum_{\substack{\alpha\neq j\\ \alpha\neq k}}\frac{(1-\rho_\alpha^2)^2(\rho_\alpha^2+\rho_j^2-2\rho_\alpha^2\rho_j^2)(\rho_\alpha^2+\rho_k^2-2\rho_\alpha^2\rho_k^2)}{(\rho_\alpha^2-\rho_j^2)(\rho_\alpha^2-\rho_k^2)}(\psi_2(x_\alpha)-\psi_1(x_\alpha)^2)$$

$$g_4 = \tilde{g}_4 + 4\sum_\alpha(1-\rho_\alpha^2)^4(P_2-2\rho_2^2)\rho_\alpha^2(1-3\rho_\alpha^2)(\psi_4(x_\alpha)-\psi_1(x_\alpha)\psi_3(x_\alpha))$$

$$+4\sum_{j\neq\alpha}\frac{(1-\rho_\alpha^2)^4\rho_\alpha^2(1-3\rho_\alpha^2)(\rho_\alpha^2+\rho_j^2-2\rho_\alpha^2\rho_j^2)}{\rho_\alpha^2-\rho_j^2}(\psi_4(x_\alpha)-\psi_1(x_\alpha)\psi_3(x_\alpha))$$

$$g_6 = \tilde{g}_6 + 8\sum_\alpha(1-\rho_\alpha^2)^6\rho_\alpha^4(1-3\rho_\alpha^2)^2(\psi_6(x_\alpha)-\psi_3(x_\alpha)^2)$$

ただし $\tilde{g}_2$, $\tilde{g}_4$, $\tilde{g}_6$ は (11) 式の g_2, g_4, g_6 において t_α^j のところを $\psi_j(x_\alpha)$ でおきかえたものとする．

系 ρ_α が単根のとき　$(1>\rho_\alpha>0)$

$$P(\sqrt{m}(r_\alpha^2-\rho_\alpha^2)/\tau_\alpha < x) = \Phi(x) - \frac{1}{\sqrt{m}}\Big(\frac{d_1}{\tau_\alpha}\Phi^{(1)}(x) + \frac{d_3}{\tau_\alpha^3}\Phi^{(3)}(x)\Big)$$

(13) $$+\frac{1}{m}\Big(\frac{g_2}{\tau_\alpha^2}\Phi^{(2)}(x) + \frac{g_4}{\tau_\alpha^4}\Phi^{(4)}(x) + \frac{g_6}{\tau_\alpha^6}\Phi^{(6)}(x)\Big) + 0\Big(\frac{1}{m\sqrt{m}}\Big)$$

ただし

$$d_1 = (P_2-2\rho_\alpha^2)(1-\rho_\alpha^2) + \sum_{j\neq\alpha}\frac{1-\rho_\alpha^2}{\rho_\alpha^2-\rho_j^2}(\rho_\alpha^2+\rho_j^2-2\rho_\alpha^2\rho_j^2)$$

$$d_3 = 4(1-\rho_\alpha^2)^3(1-3\rho_\alpha^2)\rho_\alpha^2$$

$$g_2 = \frac{1}{2}d_1^2 + (1-\rho_\alpha^2)^2\{P_2+2\rho_\alpha^2(13\rho_\alpha^2-6-2\Delta-2P_2)\}$$

(14) $$+\sum_{j\neq\alpha}\frac{(1-\rho_\alpha^2)^2}{(\rho_\alpha^2-\rho_j^2)^2}\{(\rho_\alpha^2+\rho_j^2-2\rho_\alpha^2\rho_j^2)^2-8\rho_\alpha^2\rho_j^2(1-\rho_\alpha^2)(1-\rho_j^2)\}$$

$$+\sum_{j\neq\alpha}\frac{(1-\rho_\alpha^2)^3}{\rho_\alpha^2-\rho_j^2}\left\{-4\rho_\alpha^2(\rho_\alpha^2+\rho_j^2-2\rho_\alpha^2\rho_j^2)+2\rho_j^2(1-\rho_\alpha^2)^2\right\}$$

$$g_4=d_1d_3+4\rho_\alpha^2(1-\rho_\alpha^2)^2(1-2\rho_\alpha^2)(2-13\rho_\alpha^2)$$

$$g_6=\frac{1}{2}d_3^2$$

ここで特に $x=0$ とすれば

$$P(r_\alpha^2<\rho_\alpha^2)=\frac{1}{2}-\frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{m}}\frac{1}{2\rho_\alpha}\left(P_2-1+\rho_\alpha^2+\sum_{j\neq\alpha}\frac{\rho_\alpha^2+\rho_j^2-2\rho_\alpha^2\rho_j^2}{\rho_\alpha^2-\rho_j^2}\right)+O(m^{-3/2})$$

となり最大正準相関係数については $P(r_1^2<\rho_1^2)<\frac{1}{2}+O(m^{-3/2})$ となることがわかる。

§2. 重根の場合　母正準相関係数に重根がある場合，その重複度を

$$(15)\qquad 1>\underbrace{\rho_1\cdots,\rho_1}_{k_1\text{個}}>\underbrace{\rho_2,\cdots,\rho_2}_{k_2\text{個}}>\cdots>\underbrace{\rho_\pi,\cdots,\rho_\pi}_{k_\pi\text{個}}>0$$

これに通し番号をつけて $g_0=1,\ g_1=k_1+1,\ g_2=k_1+k_2+1,\cdots g_\pi=P_1+1$ とおく。Lawley〔2〕又は藤越〔1〕により攝動法を用いて（4）に対応する展開を求めれば

$$W_\alpha=\rho_\alpha^2 I+\frac{1}{\sqrt{m}}\left\{(1-\rho_\alpha^2)^2A_{\alpha\alpha}-\rho_\alpha^2(1-\rho_\alpha^2)B_{\alpha\alpha}\right\}$$

$$(16)\qquad +\frac{1}{m}\Big[\sum_{j\neq\alpha}\frac{(1-\rho_\alpha^2)(1-\rho_j^2)}{\rho_\alpha^2-\rho_j^2}\left\{(1-\rho_\alpha^2)A_{\alpha j}-\rho_\alpha^2B_{\alpha j}\right\}\left\{(1-\rho_j^2)A_{j\alpha}-\rho_j^2B_{j\alpha}\right\}$$

$$+\sum_j(1-\rho_\alpha^2)(1-\rho_j^2)\left\{-(1-\rho_\alpha^2)A_{\alpha j}+\rho_\alpha^2B_{\alpha j}\right\}(A_{j\alpha}+B_{j\alpha})\Big]+O(m^{-3/2})$$

となり $r_1^2,\cdots,r_{P_1}^2$ の分布は $diag(W_1,\cdots,W_\pi)$ の固有根の分布と一致する。ここで $B_{\alpha j}$ は S_e を重複度 $(k_1,\cdots,k_\pi)$ に応じて分割したとき第 (α,j) 番目に表われる小行列から作られるも

の，すなわち $B_{\alpha j}=\sqrt{m}\,[S_e/m-I]_{\alpha j}$．同様に $A_{\alpha j}=\sqrt{m}\,[S_h/m-\Theta]_{\alpha j}$，$\Theta=\mathrm{diag}\,(\overbrace{\rho_1^2/(1-\rho_1^2),\cdots,\rho_1^2/(1-\rho_1^2)}^{k_1},\cdots,\overbrace{\rho_r^2/(1-\rho_r^2),\cdots,\rho_r^2/(1-\rho_r^2)}^{k_r})$ で $A_{\alpha j}$，$B_{\alpha j}$ は共に $k_\alpha\times k_j$ の行列である．これより補助定理を用いて $\sqrt{m}(W_\alpha-\rho_\alpha^2 I)_{1\leqq\alpha\leqq r}$ の特性関数を評価すれば，$\tau_\alpha=2\rho_\alpha(1-\rho_\alpha^2)$ として

$$
\begin{aligned}
&\mathrm{etr}\,(-\tfrac{1}{2}\sum_\alpha \tau_\alpha^2 T_\alpha^2)\cdot\Big[1+\frac{1}{\sqrt{m}}\Big\{i\sum_\alpha(1-\rho_\alpha^2)(P_2-\rho_\alpha^2-k_\alpha\rho_\alpha^2)\,\mathrm{tr}\,T_\alpha \\
(17)\quad &+i\sum_{j\neq\alpha}\frac{(1-\rho_\alpha^2)}{\rho_\alpha^2-\rho_j^2}(\rho_\alpha^2+\rho_j^2-2\rho_\alpha^2\rho_j^2)k_j\,\mathrm{tr}\,T_\alpha+4i^3\sum_\alpha\rho_\alpha^2(1-\rho_\alpha^2)^3(1-3\rho_\alpha^2)\,\mathrm{tr}\,T_\alpha^3\Big\}+o(m^{-1})\Big]
\end{aligned}
$$

藤越〔1〕に従ってこれを反転するには $\varphi(W_\alpha)=\mathrm{etr}\,(iT_\alpha W_\alpha-\frac{1}{2}W_\alpha^2)$ とおき正規分布に関する積分

$$
\begin{aligned}
&\frac{1}{2^{\frac{1}{2}k_\alpha}\ \pi^{k_\alpha(k_\alpha+1)/4}}\int\varphi(W_\alpha)\,dW_\alpha=\mathrm{etr}\,(-\tfrac{1}{2}T_\alpha^2) \\
(18)\quad &\qquad\quad〃\qquad\int\mathrm{tr}\,W_\alpha\cdot\varphi(W_\alpha)\,dW_\alpha=i\,\mathrm{tr}\,T_\alpha\ \mathrm{etr}\,(-\tfrac{1}{2}T_\alpha^2) \\
&\qquad\quad〃\qquad\int\{\mathrm{tr}\,W_\alpha^3-\tfrac{3}{2}(k_\alpha+1)\,\mathrm{tr}\,W_\alpha\}\,\varphi(W_\alpha)\,dT_\alpha=i^3\,\mathrm{tr}\,T_\alpha^3\ \mathrm{etr}\,(-\tfrac{1}{2}T_\alpha^2)
\end{aligned}
$$

に注意すればよい．藤越〔1〕には更にくわしい公式が述べられている．これより $V_\alpha=\sqrt{m}\,(W_\alpha-\rho_\alpha^2 I)/\tau_\alpha$ $1\leqq\alpha\leqq r$ の同時確率密度関数は次のように展開される．

$$
\begin{aligned}
(19)\quad &\prod_{\alpha=1}^{r}\frac{1}{2^{k_\alpha/2}\ \pi^{k_\alpha(k_\alpha+1)/4}}\,\mathrm{etr}\,(-\tfrac{1}{2}V_\alpha^2)\cdot\Big[1+\frac{1}{\sqrt{m}}\Big\{\sum_\alpha(1-\rho_\alpha^2)(P_2-\rho_\alpha^2-k_\alpha\rho_\alpha^2)\frac{\mathrm{tr}\,V_\alpha}{\tau_\alpha} \\
&+\sum_{j\neq\alpha}\frac{(1-\rho_\alpha^2)}{\rho_\alpha^2-\rho_j^2}(\rho_\alpha^2+\rho_j^2-2\rho_\alpha^2\rho_j^2)k_j\frac{\mathrm{tr}\,V_\alpha}{\tau_\alpha}+4\sum_\alpha\rho_\alpha^2(1-\rho_\alpha^2)^3(1-3\rho_\alpha^2)\frac{1}{\tau_\alpha^3} \\
&\qquad\cdot(\mathrm{tr}\,V_\alpha^3-\tfrac{3}{2}(k_\alpha+1)\,\mathrm{tr}\,V_\alpha)\Big\}+O(m^{-1})\Big]
\end{aligned}
$$

$\mathrm{diag}\,(V_1,\cdots,V_r)$ の固有根の分布を求めればそれが $t_j=\sqrt{m}(r_j^2-\rho_\alpha^2)/\tau_\alpha$，$j=g_{\alpha-1},\,g_{\alpha-1}+1,\,\cdots,\,g_\alpha-1$，$\alpha=1,2,\cdots,\alpha$ の同時分布を与えることになる．それには V_α をその固有根と直交行列とに分

解する変換 $V_\alpha = H_\alpha D_\alpha H_\alpha'$ を行い，その Jacobian が $\prod_{\alpha=1}^{r}\{\pi^{k_\alpha^2/2}/\Gamma_{k_\alpha}(k_\alpha/2)\}\prod_{\alpha=1}^{r}\prod_{g_{\alpha-1}\leq i<j<g_\alpha}(t_i - t_j)\,dt_1\cdots dt_{p_1}\,dH_1\cdots dH_r$ となることおよび（19）が H_α を含まぬことから直交行列上の Haar 測度 $dH_1\cdots dH_r$ で積分すればよい．ただし $g_{\alpha-1}\leqq i<j<g_\alpha$ のとき $t_i>t_j$ である．これより正準相関係数の確率密度について次の展開を得る．

<u>定理２</u>　母正準相関係数 ρ_α が重複度（15）を持つとする．標本正準相関係数 $r_1,\cdots,r_{p_1}$ について $t_j=(r_j^2-\rho_\alpha^2)\sqrt{m}/\tau_\alpha$ $\tau_\alpha=2\rho_\alpha(1-\rho_\alpha^2)$ $j=g_{\alpha-1},g_{\alpha-1}+1,\cdots,g_\alpha-1$ とおく．

$t_1,t_2,\cdots,t_{p_1}$ の確率密度関数は次の展開をもつ

$$\prod_{\alpha=1}^{r}\frac{\pi^{k_\alpha(k_\alpha-1)/4}}{2^{k_\alpha/2}\ \Gamma_{k_\alpha}(\frac{1}{2}k_\alpha)}\cdot\exp\left(-\frac{1}{2}\sum_{j=1}^{p_1}t_j^2\right)\cdot\prod_{\alpha=1}^{r}\ \prod_{g_{\alpha-1}\leq i<j<g_\alpha}(t_i-t_j)$$
$$\cdot\left[1+\frac{1}{\sqrt{m}}\left\{\sum_{\alpha}\frac{p_2-\rho_\alpha^2-k_\alpha\rho_\alpha^2}{2\rho_\alpha}\sum_{l=g_{\alpha-1}}^{g_\alpha-1}t_l+\sum_{\alpha\neq j}\frac{\rho_\alpha^2+\rho_j^2-2\rho_\alpha^2\rho_j^2}{2\rho_\alpha(\rho_\alpha^2-\rho_j^2)}k_j\sum_{l=g_{\alpha-1}}^{g_\alpha-1}t_l\right.\right.$$
$$\left.\left.+\frac{1}{2}\sum_{\alpha}\frac{1-3\rho_\alpha^2}{\rho_\alpha}\sum_{l=g_{\alpha-1}}^{g_\alpha-1}\left\{t_l^3-\frac{3}{2}(k_\alpha+1)t_l\right\}\right\}+O\left(\frac{1}{m}\right)\right]$$

特に $k_1=\cdots=k_r=1$ とすれば単根の場合となり分布関数に直せば定理１の $O(m^{-1/2})$ の項迄と一致する．重根のある場合分布関数の形で展開を陽に書くことがむつかしいことがわかる．

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