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APPROXIMATIONS FOR THE DISTRIBUTIONS OF THE EXTREME ROOTS OF FOUR DETERMINANTAL EQUATIONS IN MULTIVARIATE ANALYSIS

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§1. INTRODUCTION AND SUMMARY

Simple approximations are presented for the distribution of the extreme roots of four determinantal equations in multivariate analysis. We consider the extreme latent roots of three matrices, (i) $S_1 S_2^{-1}$ where $n_1 S_1$ and $n_2 S_2$ are independently distributed as Wishart $W_m(n_1, \Sigma_1)$ and $W_m(n_2, \Sigma_2)$ respectively, (ii) $S_1 S_2^{-1}$ where $n_1 S_1$ and $n_2 S_2$ are independently distributed as noncentral Wishart $W_m(n_1, \Sigma, \Omega)$, Ω noncentrality matrix, and $W_m(n_2, \Sigma)$ respectively and (iii) $\Sigma^{-1} S$ where nS is distributed as noncentral Wishart $W_m(n, \Sigma, \Omega)$, and (iv) the extreme canonical correlation coefficients. The approximations for cases (i), (ii) and (iii) take the form of upper and lower bounds for the distribution functions of the largest and smallest latent roots respectively, and the approximations for case (iv) are valid for large sample size. The approximations are expressed in terms of products of (i) F, (ii) noncentral F, (iii) noncentral χ^2 and (iv) noncentral F and also normal probabilities.

§2. $S_1 S_2^{-1}$; CASE (i)

Let $l_1 \geq l_2 \geq \cdots \geq l_m > 0$ be the latent roots of $S_1 S_2^{-1}$. Let A be an $m \times m$ nonsingular matrix such that

$$A \Sigma_2 A' = I_m \quad \text{and} \quad A \Sigma_1 A' = \Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \cdots, \lambda_m)$$

where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m > 0$ are the latent roots of $\Sigma_1 \Sigma_2^{-1}$. Putting $S_i^* = A S_i A'$ $(i = 1, 2)$, it follows that $n_1 S_1^*$ and $n_2 S_2^*$ are independently distributed as $W_m(n_1, \Lambda)$ and $W_m(n_2, I_m)$ respectively, and $l_1, l_2, \cdots, l_m$ are the latent roots of $S_1^* S_2^{*-1}$. It is well-known (Roy [8]) that

$$l_1 \geq \frac{\underset{\sim}{x}' S_1^* \underset{\sim}{x}}{\underset{\sim}{x}' S_2^* \underset{\sim}{x}} \geq l_m, \qquad \underset{\sim}{x}' S_2^* \underset{\sim}{x} > 0.$$

Hence, if we let $S_i^* = (s_{kl}^{(i)})$ $(i = 1, 2)$ it follows that

$$(1) \qquad l_1 \geq \max\left(\frac{s_{11}^{(1)}}{s_{11}^{(2)}}, \cdots, \frac{s_{mm}^{(1)}}{s_{mm}^{(2)}} \right)$$

and

(2) $\quad l_m \leq \min\left(\dfrac{s_{11}^{(1)}}{s_{11}^{(2)}}, \cdots, \dfrac{s_{mm}^{(1)}}{s_{mm}^{(2)}}\right).$

Now, $n_1 s_{ii}^{(1)}/\lambda_i$ and $n_2 s_{ii}^{(2)}$ $(i=1,2,\cdots,m)$ are all independently distributed as $\chi^2_{n_1}$ and $\chi^2_{n_2}$ respectively; hence the $s_{ii}^{(1)}/\lambda_i s_{ii}^{(2)}$ $(i=1,2,\cdots,m)$ have independent F_{n_1,n_2} distributions. Thus using (1) and (2) we obtain the following

<u>Theorem 1.</u> Upper and lower bounds for the distribution functions of l_1 and l_m are respectively given by

(3) $\quad P(l_1 \leq x) \leq \prod_{i=1}^{m} P\left(F_{n_1,n_2} \leq \dfrac{x}{\lambda_i}\right)$

and

(4) $\quad P(l_m \leq x) \geq 1 - \prod_{i=1}^{m} P\left(F_{n_1,n_2} \geq \dfrac{x}{\lambda_i}\right).$

The bounds are clearly exact when $m=1$, and when $\Lambda = I_m$, i.e. $\Sigma_1 = \Sigma_2$, they agree with bounds given by Mickey [4].

§ 3. $S_1 S_2^{-1}$; CASE (ii)

We can write S_1 as $n_1 S_1 = YY'$, where Y is an $m \times n_1$ matrix whose columns are independently distributed as normal with covariance matrix Σ and $E(Y) = M$, and $\Omega = \Sigma^{-1}MM'$. Let $l_1 \geq l_2 \geq \cdots \geq l_m > 0$ be the latent roots of $S_1 S_2^{-1}$. Let A be an $m \times m$ nonsingular matrix such that

$$A\Sigma A' = I_m \quad \text{and} \quad AMM'A' = \Omega_D = \text{diag}\,(w_1, w_2, \cdots, w_m)$$

where $w_1 \geq w_2 \geq \cdots \geq w_m \geq 0$ are the latent roots of $\Sigma^{-1}MM' = \Omega$. Putting $S_i^* = AS_iA'$ $(i = 1, 2)$ we have that $n_1 S_1^*$ and $n_2 S_2^*$ are independently distributed as $W_m(n_1, I_m, \Omega_D)$ and $W_m(n_2, I_m)$ respectively, and $l_1, l_2, \cdots, l_m$ are the latent roots of $S_1^* S_2^{*-1}$. Put $S_i^* = (s_{kl}^{(i)})$ $(i = 1, 2)$; it follows that $n_1 s_{ii}^{(1)}$ and $n_2 s_{ii}^{(2)}$ are independently distributed as noncentral $\chi^2_{n_1}(w_i)$ with noncentrality parameter w_i and $\chi^2_{n_2}$ respectively. Hence the $s_{ii}^{(1)} / s_{ii}^{(2)}$ have independent noncentral $F_{n_1, n_2}(w_i)$ distributions. Thus, using (1) and (2) we obtain the following

<u>Theorem 2.</u> Upper and lower bounds for the distribution functions of l_1 and l_m are respectively given by

(5) $\quad P(l_1 \leq x) \leq \prod_{i=1}^{m} P(F_{n_1,n_2}(w_i) \leq x)$

and

(6) $\quad P(l_m \leq x) \geq 1 - \prod_{i=1}^{m} P(F_{n_1,n_2}(w_i) \geq x).$

Numerical examinations showed that the bounds (3) and (5) appear quite reasonable as quick approximations to the exact probabilities.

§4. $\Sigma^{-1}S$; CASE (iii)

Let $l_1 \geq l_2 \geq \cdots \geq l_m > 0$ be the latent roots of $\Sigma^{-1}S$. As in Section 3, let A be an $m \times m$ nonsingular matrix such that

$$A\Sigma A' = I_m \quad \text{and} \quad AMM'A' = \Omega_D = \mathrm{diag}(w_1, w_2, \cdots, w_m)$$

where $w_1 \geq w_2 \geq \cdots \geq w_m \geq 0$ are the latent roots of Ω $(= \Sigma^{-1}MM')$. Then $nS^* = nASA'$ has the $W_m(n, I_m, \Omega_D)$ distribution and $l_1, l_2, \cdots, l_m$ are the latent roots of S^*. Now it is well-known (Bellman [1], P111) that

$$l_1 \geq \frac{\underset{\sim}{x}'S^*\underset{\sim}{x}}{\underset{\sim}{x}'\underset{\sim}{x}} \geq l_m$$

and hence that

$$l_1 \geq \max(s_{11}, \cdots, s_{mm})$$

and

$$l_m \leq \min(s_{11}, \cdots, s_{mm})$$

where $S^* = (s_{ij})$. These inequalities, together with the fact that the ns_{ii} $(i = 1, 2, \cdots, m)$ have independent $\chi^2_n(w_i)$ distributions, yield the following

Theorem 3. Upper and lower bounds for the distribution functions of l_1 and l_m are respectively given by

(7) $$P(l_1 \leq x) \leq \prod_{i=1}^{m} P(\chi^2_n(w_i) \leq nx)$$

and

(8) $$P(l_m \leq x) \geq 1 - \prod_{i=1}^{m} P(\chi^2_n(w_i) \geq nx).$$

The bounds are exact when $m = 1$ and, when $\Omega_D = 0$, i.e. nS^* is $W_m(n, I_m)$, they agree with bounds given by Muirhead [5]. An approximation, valid for large n, to $P(l_1 \leq x)$ somewhat similar to (7) has been given by Sugiyama [9] in terms of central χ^2 probabilities.

§5. CANONICAL CORRELATION COEFFICIENTS ; CASE (iv)

Let $\rho_1 \geq \rho_2 \geq \cdots \geq \rho_m \geq 0$ and $r_1 \geq r_2 \geq \cdots \geq r_p > 0$ denote respectively the population canonical correlation coefficients and the sample canonical correlation coefficients, formed from a sample of size $n+1$. We derive simple approximations for the distribution functions of r_1^2 and r_p^2 in the case when $1 > \rho_1 > \cdots > \rho_p > 0$. Using the results in Section 3, a representation of canonical correlation coefficient, conditional on the samples on one vector (Constantine [2]) and the asymptotic expansions, in terms of normal distributions, for the distributions of the latent roots of a Wishart matrix (Muirhead and Chikuse [6]), we can obtain our results. Due to the limitation of space, we only summarize the results in

<u>Theorem 4.</u> Approximations for the distribution functions of r_1^2 and r_p^2, when $1 > \rho_1 > \rho_2 > \cdots > \rho_p > 0$, are given for large n by

(19) $P(r_1^2 \leq x) \doteqdot \prod_{i=1}^{p} P\left(F_{q,n-q}\left(n\rho_i^2(1-\rho_i^2)^{-1}\right) \leq (n-q)q^{-1}x(1-x)^{-1}\right]$

and

(10) $P(r_p^2 \le x) \doteqdot 1 - \prod_{i=1}^{p} P[F_{q,n-q}(n\rho_i^2(1-\rho_i^2)^{-1}) \ge (n-q)q^{-1}x(1-x)^{-1}]$.

Alternative approximations are given for large n by

(11) $P(r_1^2 \le x) \doteqdot \prod_{i=1}^{p} P(H_i \le x)$

and

(12) $P(r_p^2 \le x) \doteqdot 1 - \prod_{i=1}^{p} P(H_i \ge x)$,

where H_i denotes a random variable distributed as normal $N(\rho_i^2, 4\rho_i^2(1-\rho_i^2)^2 n^{-1})$, and furthermore by

(13) $P(r_1^2 \le x) \doteqdot P(H_1 \le x)$

and

(14) $P(r_p^2 \le x) \doteqdot P(H_p \le x)$.

We note that the approximate distributions $N(\rho_i^2, 4\rho_i^2(1-\rho_i^2)^2 n^{-1})$ of r_i^2, $i=1, p$, for large n, given in (13) and (14), are in fact the limiting distributions of r_i^2, $i=1, p$, derived as a special case from results due to Hsu [3].

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