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On The Homotopy Type Of Some Subgroups Of $\mathrm{Diff}(M^3)$

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Introduction

Let M be a closed oriented n-dimensional manifold and F be a codimension one foliation on M of class $C^r (r \geqq 2)$. (M,F) is called a generalized Reeb foliated manifold if (M,F) is decomposed as $(M,F) = \bigcup_{i=1}^{r} (M_i, F_i)$, where (M_i, F_i) is a generalized Reeb component for each i (see §1 for definition).

The main purpose of this paper is to show that the topological group of foliation preserving diffeomorphisms of a generalized Reeb foliated 3-dimensional manifold (M,F) has the same homotopy type as an ℓ-dimensional torus T^ℓ for some non-negative integer ℓ which can be controlled by a geometrical data(see Theorem 4.2). The key of the proof is the fibration lemma(Lemma 1.13) which is valid in the general dimensions. A typical example of generalized

Reeb foliated manifolds is constructed from a spinnable structure by the usual method (Tamura[12]). In this case we have a better information, that is, the integer ℓ is less than the number of connected components of the axis of this spinnable structure plus two (Theorem 5.2).

§1, Generalized Reeb foliation and fibration lemma

Let M be a closed oriented n-dimensional manifold and F a codimension one foliation on M of class $C^r (r \geq 2)$.

Definition 1.1. An orientation preserving diffeomorphism $f: M \longrightarrow M$ is called a foliation preserving diffeomorphism(resp. a leaf preserving diffeomorphism) if for each point x of M, the leaf through x is mapped to the leaf through $f(x)$(resp. x), that is $f(L_x)=L_{f(x)}$ (resp. $f(L_x)=L_x$), where L_x is the leaf that contains x. It is clear that a foliation preserving diffeomorphism(resp. a leaf preserving diffeomorphism) f induces a homeomorphism $\bar{f}$

(resp. id.) of the leaf space M/F such that the diagram commutes,

$$\begin{array}{ccc} M & \xrightarrow{f} & M \\ \downarrow & & \downarrow \\ M/F & \xrightarrow{\bar{f}} & M/F \end{array} \quad \left(\text{resp.} \quad \begin{array}{ccc} M & \xrightarrow{f} & M \\ \downarrow & & \downarrow \\ M/F & \xrightarrow{id.} & M/F \end{array} \right),$$

where vertical arrows are canonical projection(see Reeb[9]).

Let $FDiff^r(M,F)$ or $FDiff(M,F)$ (resp. $\widetilde{LDiff}^r(M,F)$ or $\widetilde{LDiff}(M,F)$) denote the space of all foliation(resp. leaf) preserving diffeomorphisms of (M,F) of class C^r. It is clear that $\widetilde{LDiff}(M,F) \subset FDiff(M,F) \subset Diff(M)$. Topologies of the spaces are induced by the C^r topology of $Diff(M)$. Then it is well known that these spaces are topological groups. There is an exact sequence of topological groups; $1 \to \widetilde{LDiff}(M,F) \to FDiff(M,F) \xrightarrow{\pi} Homeo(M/F)$, where the second arrow is the inclusion map and the map π is defined by $\pi(f)=\bar{f}$.

Definition 1.2. A compact foliated manifold $(M,F)(\partial M \neq \emptyset)$ is called a generalized Reeb component if the following three conditions are satisfied; (1) all leaves in Int M are non-compact and proper, (2) the holonomy groups of all leaves in Int M are trivial and (3) each of the elements of the holonomy group of each compact leaf of F can be represented by a local diffeomorphism of $R_+=[0,\infty)$, leaving fixed 0, which is C^r-tangent to id. at 0 and whose second derived function is non-negative or non-positive in some neighborhood of 0.

The structure of a generalized Reeb component was studied by Imanishi-Yagi[6]. Our definition is slightly different from that in[6]. A generalized Reeb component in [6] means a compact foliated manifold $(M,F)(\partial M\neq\emptyset)$ satisfying above (1),(2). In the first part of this section, we recall some properties of a generalized Reeb component. See [6;§2] for more details.

Definition 1.3. A vector field X on M transverse to F is called a nice vector field if X has a closed orbit C such that $C\cap L=\{\text{one point}\}$ for any leaf L in Int M. Such a closed orbit C is called a nice orbit.

Proposition 1.4 [6; Proposition 2.1]. Let (M,F) be a generalized Reeb component. Then there exists a nice vector field X on M.

We identify S^1 with the nice orbit C in Proposition 1.4. Let $p:\text{Int } M \longrightarrow S^1$ be a map defined by $p(x)=C\cap L_x$. Then we see that p is a locally trivial fibration. Let dt be the natural one form on $S^1=R^1/Z$ and $w=p^*dt$. Then there exists a positive function g on Int M such that $w(gX)\equiv 1$. Let ϕ_t denote the flow associated to gX. ϕ_t is the foliation preserving flow on Int M and $\phi_n(L)=L$ for any leaf L in Int M and any integer n.

Remark 1.5. By putting $\phi_t(z)=z$ for $z\in\partial M$, we may show from Lemma 1.8 below and Definition 1.2(3), that ϕ_t is a foliation preserving flow of class C^r on M and is C^r-tangent to id. at ∂M.

Lemma 1.6[6;Lemma 2.5]. Let V be a component of M and z a point of V. Let T be the maximal solution curve of X which contains z and y_0 be a point of $L_{x_0} \cap T$. Then $L_{x_0} \cap T = \{y_n = \phi_n(y_0),\ n \in Z\}$, and if X is outward normal at z, $\lim_{n \to \infty} y_n = z$.

To described the structure of F near V, we define a foliated manifold $V(N,h)$ as follows. Let N be a codimension one submanifold of V such that $V-N$ is connected and the manifold V_N obtained from V by cutting along N has two boundary components N_1 and N_2 which are copies of N. Let h be a contracting diffeomorphism of $[0,\varepsilon]$, $\varepsilon > 0$. $V(N,h)$ is obtained from $V_N \times [0,\varepsilon)$ by identifying $(x,t) \in N_1 \times [0,\varepsilon)$ with $(x,h(t)) \in N_2 \times [0,\varepsilon)$. There exists a dually foliated structure on $V(N,h)$ which is induced from the product structure $V_N \times [0,\varepsilon)$. The dual structure of F is defined by X.

Lemma 1.7[6;Lemma 2.6]. There exist a submanifold N and a diffeomorphism h satisfying above conditions. There exists an embedding j of $V(N,h)$ into M which preserves the dually foliated structures, satisfying $j(x,0)=x$ for $x\in V$.

Lemma 1.8[6; Lemma 2.7]. Let $j\colon V(N,h)\longrightarrow M$ be as above. We identify $(x,\tau)\in(V_N-N_2)\times[0,\varepsilon)$ with a point of $V(N,h)$. For $t\geq 0$ we define $\phi_t'(x,\tau)=j^{-1}\circ\phi_t\circ j(x,\tau)$, then ϕ_t' preserves the foliated structure on $V(N,h)$ and we have $\phi_{\ell k}'(x,\tau)=(x,h^{\ell}(\tau))$.

For any f in $FDiff(M,F)$, there exists a diffeomorphism $\bar{f}$ of S^1 such that the diagram commutes,

$$\begin{array}{ccc} \mathrm{Int}\,M & \xrightarrow{f|\mathrm{Int}\,M} & \mathrm{Int}\,M \\ \downarrow p & & \downarrow p \\ S^1 & \xrightarrow{\quad \bar{f} \quad} & S^1 \end{array} .$$

Let $FDiff_0(M,F)$ be the identity component of $FDiff(M,F)$. Let $\pi\colon FDiff_0(M,F)\longrightarrow Diff_0(S^1)$ is a map defined by $\pi(f)=\bar{f}$. clealy this map is the continuous homomorphism.

Lemma 1.9. $\mathrm{Im}\,\pi = SO(2)$.

Proof. That $\mathrm{Im}\,\pi$ contains the rotation group $SO(2)$, is easily proved by using the foliation preserving flow ϕ_t. Let us prove that $\mathrm{Im}\,\pi$ is contained in $SO(2)$. Suppose for some f in $FDiff_0(M,F)$, $\pi(f) \notin SO(2)$. We shall deduce a contradiction from this assumption.

A point x_0 in the nice orbit C corresponds to a point $\bar{x}_0$ in S^1. We can assume $\pi(f)(\bar{x}_0)=f(\bar{x}_0)=\bar{x}_0, f \neq id.$, by composing a relevant rotation which is induced by the foliation preserving flow ϕ_t.

Assertion 1.10. There exists a leaf preserving diffeomorphism g such that $g f$ preserves each orbit of X in some small neighborhood of ∂M.

Proof. Let V be a component of ∂M and $f^{-1}|_V : V \longrightarrow V$ be the diffeomorphism restricted to V of f^{-1}, which is contained in the identity component of the space of diffeomorphisms of V, $\mathrm{Diff}_0(V)$. Take a smooth path h_t from id_V to $f^{-1}|_V$ in $\mathrm{Diff}_0(V)$, $h_0=id_V$, $h_1=f^{-1}|_V$. Let $H:V\times I\to V\times I$ be a map defined by $H(x,t)=(h_t(x),t)$. Consider a vector field defined by $(\frac{\partial h}{\partial t}t,1)$ on $V\times I$ in $M\times I$. Take small tubular neighborhoods N_1,N_2 of $V\times I$ in $M\times I$, $N_1\supset\overline{N_2}$ and the vector field, which is denoted by $V(x,t)$, on $M\times I$ such that it is tangent to the leaves and the derivative dp_1 of the projection $p_1:M\times I\to M$ maps $v(x,t)$ to the zero vector outside N_1, and the derivative dp_2 of the projection $p_2:M\times I\to I$ maps $v(x,t)$ to the unit vector $\frac{\partial}{\partial t}$, and that in N_2 it commutes with the differential map of the projection of N_1 to V along the orbits of X. Then integrating the vector field $v(x,t)$, we obtain an element g_1 of $\mathrm{LDiff}(M,F)$ which is the extention of $f^{-1}|_V$. Note that $\pi(g_1\circ f)=\overline{f}$ and $g_1\circ f|_V=id_V$. By composing a relevant leaf preserving diffeomorphism g_2 such that $g_2|_V=id_V$ and $g_2|_{(\text{outside of } N_2)}=id.$, $g_2\circ g_1\circ f$

has a required property. It is similar for the case of other component of ∂M. Q.E.D.

Again we denote such $g\circ f$ by f for simplicity.

Assertion 1.11. Under Lemma 1.6, there exists a unique integer m such that $f(y_n)=y_{n+m}$ for a sufficiently large integer n.

Proof. From Assertion 1.10, there is a commutative diagram in T,

$$\begin{array}{ccc} [y_n, y_{n+1}] & \xrightarrow{\ f\ } & [y_{n+m}, y_{n+m'}] \\ \downarrow p & & \downarrow p \\ S^1 & \xrightarrow{\ \bar{f}\ } & S^1 \end{array},$$

where $p(y_n)=x_0$ for each n and $[y_n, y_{n+1}] = \bigcup_{0\leq t\leq 1} \phi_t(y_n)$ in T. Therefore $m'=m\pm 1$. Since $\bar{f}$ is the orientation preserving diffeomorphism, we have $m'=m+1$. Q.E.D.

Proof of Lemma 1.9 cntinued. By composing the foliation preserving diffeomorphism induced from Φ_{-m}, we may assume $f(y_n)=y_n$ for a large positive integer n. Let T_n denote a set

$\left\{\bigcup_{t\geq 0}\Phi_t(y_n)\right\}\cup\{z\}$ and $f|_{T_n}:T_n\to T_n$ be the restriction of f to T_n. We can assume that T_n is parametrized by the interval $[0,\varepsilon]$ such that z corresponds to 0. Put $f_0=f|_{[y_n,y_{n+k}]}$.

The diffeomorphism $f|_{T_n}$ is described by f_0 as follows;

$$f(x)=\begin{cases} h^{\ell}\circ f_0\circ h^{-\ell}(x)\ , & \text{for } x\in[y_{n+(\ell-1)k},y_{n+\ell k}]\ , \\ x\ , & \text{for } x=z\ , \end{cases}$$

where h, which is that in Lemmas 1.7 and 1.8, is a contracting diffeomorphism of $T_n\cong[0,\varepsilon]$. Note that the second derived function $h''\leq 0$ in some neighborhood of 0 from Definition 1.2(3). From the assumption, $f_0\neq$ id., there is $x_0\in[y_n,y_{n+k}]$ that satisfies the following 1) or 2) ;

1) $x_0\geq f_0(x_0)$ and $f_0'(x_0)>1$,

2) $x_0\leq f_0(x_0)$ and $f_0'(x_0)<1$.

Let $x_n=h^n(x_0)$ $(n=1,2,\ldots)$. When x_0 satisfies the condition 1),

$$f'(x_n)=\frac{(h^n)'(f_0(x_0))}{(h^n)'(x_0)}f_0'(x_0)$$

$$\geqq f_0'(x_0)>1 .$$

Hence $f'(x_n)$ can not converge to 1. This fact and $f(y_n)=y_n$ lead to a contradiction. It is similarly proved when x_0 satisfies the condition 2). Q.E.D.

Definition 1.12. F is called a generalized Reeb foliation on a closed oriented manifold M if there is a decomposition of (M,F) such that $(M,F)=\bigcup_{i=1}^{r}(M_i,F_i)$, where (M_i,F_i) denotes a generalized Reeb component.

Let f_i be the restriction of f to (M_i,F_i) for any f in $FDiff_0(M,F)$. From Lemma 1.9, we define a map $\pi : FDiff_0(M,F) \longrightarrow \overbrace{SO(2)\times\cdots\times SO(2)}^{r}$ by $(f)=(\bar{f}_1,\ldots,\bar{f}_r)$.

Lemma 1.13(fibration lemma). π is a locally trivial fibration.

Proof. We define a foliation preserving flow ϕ_t on M to be a union of the foliation preserving flows on generalized Reeb components. From Remark 1.5, Φ_t is well defined and of class C^r. Hence it is easily proved by using this flow Φ_t. Q.E.D.

Let LDiff(M,F) denote the fiber of this fibration . Note that this space is the space $\widetilde{\mathrm{LDiff}}(M,F) \cap \mathrm{FDiff}_0(M,F)$.

Corollary 1.14. $\mathrm{FDiff}_0(M,F)/\mathrm{LDiff}(M,F)$ is homeomorphic to $\overbrace{S^1 \times \dots \times S^1}^{r}$.

Let $\mathrm{LDiff}_0(M,F)$ denote the identity component of LDiff(M,F). Since LDiff(M,F) is a closed subgroup of $\mathrm{FDiff}_0(M,F)$ and the natural map $\mathrm{FDiff}_0(M,F) \longrightarrow \mathrm{FDiff}_0(M,F)/\mathrm{LDiff}(M,F)$ has a local section, we use "the bundle structure theorem"(Steenrod[11;p30]).

Proposition 1.15. Let $p : \mathrm{FDiff}_0(M,F)/\mathrm{LDiff}_0(M,F) \longrightarrow \mathrm{FDiff}_0(M,F)/\mathrm{LDiff}(M,F)$ be the map induced by the inclusion of cosets. Then we can assign a bundle structure to $\mathrm{FDiff}_0(M,F)/\mathrm{LDiff}_0(M,F)$ relative to p. The fiber of the bundle is

LDiff(M,F)/$LDiff_0$(M,F).

Corolłary 1.16. $FDiff_0(M,F)/LDiff_0(M,F)$ is homeomorphic to an r-dimensional manifold which has the same homotopy type as an ℓ-dimensional torus $T^\ell (0 \leq \ell \leq r)$.

Remark 1.17. Leslie[7] has proved " let (M,F) be a compact foliated n-dimensional manifold of codimension q, and of class C^∞. If F has a finite number of leaves $L_1, \ldots, L_\ell$ such that $\overline{L_1 \cup \ldots \cup L_\ell} = M$, then $FDiff_0(M,F)/LDiff_0(M,F)$ is a Lie group of dimension $\leq \ell k$.

§2 On the space LDiff(M,F)

Let (M,F) be a generalized Reeb foliated manifold and V_i $(i=1,2,\ldots,\lambda)$ its compact leaves. Let $\mathfrak{L}$ denote the subspace of LDiff(M,F) consisting of leaf preserving diffeomorphisms such that in some tubular neighborhood $N(V_i)$ of V_i, the following diagram commutes;

$$\begin{array}{ccc} N(V_1)\cup\ldots\cup N(V_\lambda) & \xrightarrow{f} & f(N(V_1)\cup\ldots\cup N(V_\lambda))\subset M \\ \downarrow q & & \downarrow q \\ V_1\cup\ldots\cup V_\lambda & \xrightarrow{f|V_1\cup\ldots\cup V_\lambda} & V_1\cup\ldots\cup V_\lambda , \end{array}$$

where $q : N(V_i)\longrightarrow V_i$ is a map defined by $q(y)=\lim_{t\to\infty}\phi_t(y)$ (see Lemma 1.6).

Lemma 2.1. The inclusion map £$\hookrightarrow$LDiff(M,F) is a weak homotopy equivalence.

£ is included in $\mathrm{FDiff}_0(M,F)$, hence the restriction to each V_i belong to the identity component $\mathrm{Diff}_0(V_i)$ of $\mathrm{Diff}(V_i)$. Let res : £$\longrightarrow \mathrm{Diff}_0(V_1)\times\ldots\times\mathrm{Diff}_0(V_\lambda)$ be the restriction map, i.e., $\mathrm{res}(f)=(f|_{V_1},\ldots,f|_{V_\lambda})$.

Lemma 2.2. There is an exact sequence;

$$1\longrightarrow \mathcal{G}\longrightarrow £ \xrightarrow{res} \mathrm{Diff}_0(V_1)\times\ldots\times\mathrm{Diff}_0(V_\lambda)\longrightarrow 1 ,$$

where $\mathcal{G}$ is the kernel of res , and res is a locally trivial fibration.

proof. This is proved by the same way as in [4;Lemma 3] .

Let (M_1, F_1) and (M_2, F_2) be generalized Reeb components with V_i as a component of boundary. For f in $\mathcal{J}$, $\rho_i(f)$ is a pair of integers (k, k'), where k and k' are the integer m in Assertion 1.11 for $(M_1, F_1), (M_2, F_2)$ respectively.

Lemma 2.3. $\rho_1 \oplus \ldots \oplus \rho_\lambda : \mathcal{J} \to (Z \oplus Z) \oplus \ldots \oplus (Z \oplus Z)$ is a homomorphism.

Proof. It is easily proved by the following commutative diagram;

$$\begin{array}{ccccc}
[y_n, y_{n+1}] & \xrightarrow{f} & [y_{n+m}, y_{n+m+1}] & \xrightarrow{g} & [y_{n+m+\ell}, y_{n+m+\ell+1}] \\
\downarrow p & & \downarrow p & & \downarrow p \\
S^1 & \xrightarrow{\bar{f}} & S^1 & \xrightarrow{\bar{g}} & S^1
\end{array},$$

wher n is a sufficiently large positive integer.

Remark 2.4. Clearly $\rho_1 \oplus \ldots \oplus \rho_\lambda$ is a locally trivial fibration over the image of $\rho_1 \oplus \ldots \oplus \rho_\lambda$.

§3. The homotopy type of the space of diffeomorphisms of 2-dimensional manifold and its application.

Let M_g be a closed oriented 2-dimensional manifold of genus g and $D_1^2 \cup D_2^2 \cup \ldots \cup D_\ell^2$ be 2-discs embedded in M_g. Let $\mathrm{Diff}^r(M_g)$

be the space of orientation preserving diffeomorphisms of M_g of class C^r with C^r-topology and $\mathrm{Diff}_0(M_g)$ be its identity component. By $\mathrm{Diff}^r(M_g;D_1\cup\ldots\cup D_\ell)$ we denote the subgroup of $\mathrm{Diff}^r(M_g)$ consisting of the diffeomorphisms whose restriction to $D_1\cup\ldots\cup D_\ell$ are identity.

Proposition 3.1. $\mathrm{Diff}^r_0(M_g;D_1\cup\ldots\cup D_\ell)$ is contractible for any g and any positive integer ℓ.

Lemma 3.2. Let V be a compact oriented 2-dimensional manifold with boundary. Then $\mathrm{res}:\mathrm{Diff}^r_0(V)\longrightarrow\mathrm{Diff}^r_0(\partial V)$ is a locally trivial fibration, where res is the restriction map.

Proof. It is easy to see that res is surjective. Let $U(id.)$ be a neighborhood of id. in $\mathrm{Diff}^r_0(\partial V)$. We may consider $U(id.)$ as the set consisting of sections s of the tangent bundle $T(\partial V)$ of ∂V such that the norm of s, $\|s\|<\varepsilon$ for a small positive number ε. To prove Lemma 3.2, in the following diagram

$$\begin{array}{ccc} T(\partial V) & \hookrightarrow & T(V) \\ s\uparrow\downarrow & & S\,\vdots\!\uparrow\downarrow \\ \partial V & \hookrightarrow & V \end{array},$$

we have only extend the section s of $T(\partial V)$ to $T(V)$. Let N be a tubular neighborhood of ∂V in V, which is diffeomorphic to $\partial V \times [0,1)$. Since $T(V)|_N = N \times R^2$, for any section s in $U(\mathrm{id.})$, we define a section of $T(V)$ $S: N \longrightarrow T(V)|_N$ by $S(v,t) = \{(v,t), \chi(t)s(v), 0\}$, where $\chi: [0,1) \longrightarrow [0,1]$ is a smooth function such that $\chi[0,1/3]=1$, $\chi[2/3,1)=0$. Q.E.D.

Proof of Proposition 3.1. Let V be a compact oriented 2-dimensional manifold(with or without boundary) which is not diffeomorphic to a 2-sphere S^2, a 2-torus T^2, a 2-disc D^2 and a cylinder $C^2(=S^1 \times [0,1])$. The group $\mathrm{Diff}^r_0(V)$ is contractible(see Gramain [5]). Note that the fiber of the fibration in Lemma 3.2 is $\mathrm{Diff}^r_0(V;\partial V)$. Hence $\mathrm{Diff}^r_0(V;\partial V)$ is contractible. It is well known that $\mathrm{Diff}_0^r(D^2;\partial D^2)$ is contractible(see Smale [10]). For the case of $V=C^2$, we easily see that $\mathrm{Diff}_0^r(C^2;\partial C^2)$ is contractible.

Q.E.D.

Next, we consider the non-compact case. Let L be a non-compact oriented 2-dimensional manifold. By $\mathrm{Diff}^{c,r}(L)$ we denote the subgroup of $\mathrm{Diff}^r(L)$ consisting of diffeomorphisms with compact support.

Proposition 3.3. $\pi_i(\mathrm{Diff}^{c,r}(L);\mathrm{id.})=0$ for each positive integer i.

Proof. Let S^i be a i-dimensional sphere with a base point $s_0 (i\geq 1)$. Let $\varphi:(S^i,s_0)\longrightarrow(\mathrm{Diff}^c(L),\mathrm{id.})$ be any continuous map. Since S^i is compact, there exists a compact submanifold K of L such that $\varphi(S^i)$ restricted to $L-K$ is identity. Hence the image of φ is contained in $\mathrm{Diff}(K;\partial K)$. From the contractibility of the identity component of $\mathrm{Diff}(K;\partial K)$, there exists a homotopy

$\Phi: S^1 \times I \to \mathrm{Diff}(K;\partial K)$ such that $\Phi(s,0)=\varphi(s)$ and $\Phi(s,1)=id.$.

Q.E.D.

Let E^3 be the total space of a fibration over S^1 with L^2 as a fiber, that is, $E=L\times I/(x,0)\sim(h(x),1)$, where L is a non-compact oriented 2-dimensional manifold and $h:L\to L$ is an orientation preserving diffeomorphism. Then we study the homotopy type of the space $\{f\in \mathrm{Diff}^c(E); \pi\circ f=\pi$, where π is the fibration map$\}$, denoted by $P^h\mathrm{Diff}^c(L))$. This space is identified with the space $\{\varphi: I\to \mathrm{Diff}^c(L)$, differentiable map; $\varphi(0)=h^{-1}\circ\varphi(1)\circ h\}$. Furthermore, this space is homotopy equivalent to the space $\{\varphi: I\to \mathrm{Diff}^c(L)$, continuous map; $\varphi(0)=h^{-1}\circ\varphi(1)\circ h\}$ with C-O topology, which is also denoted by $P^h(\mathrm{Diff}^c(L))$. Let $q:P^h(\mathrm{Diff}^c_0(L))\to \mathrm{Diff}^c_0(L)$ be a map defined by $q(\varphi)=\varphi(0)$.

Lemma 3.4. q is a locally trivial fibration.

Proof. First we show q is surjective. For any f in

$\mathrm{Diff}_0^c(L)$, take a smooth path f_t from identity to f in $\mathrm{Diff}_0^c(L)$.

$h^{-1}\circ f_t\circ h$ is a smooth path from identity to $h^{-1}\circ f\circ h$ in $\mathrm{Diff}_0^c(L)$.

Let g_t be a homotopy defined by

$$g_t = \begin{cases} f_{1-2t} & \text{for } 0 \leqq t \leqq \frac{1}{2}, \\ h^{-1}\circ f_{2t-1}\circ h & \text{for } \frac{1}{2} \leqq t \leqq 1. \end{cases}$$

g_t is a path connecting f and $h^{-1}\circ f\circ h$.

Next, we show that q is a locally trivial fibration. Let $U_\varepsilon(\mathrm{id.})$ be a neighborhood of id. in $\mathrm{Diff}_0^c(L)$, which is homeomorphic to the set $\{s\in\Gamma_c(T(L));\ \|s\|<\varepsilon\}$ (by a coordinate mapping (Eells[3])), where $\Gamma_c(T(L))$ is the space of sections of the tangent bundle $T(L)$ of L whose restrictions to outside of the compact set are zero-sections. Because of the continuity of a map $f \longrightarrow h^{-1}\circ f\circ h$, there exists a neighborhood $U_\delta(\mathrm{id.})$ such that for any f in $U_\delta(\mathrm{id.})$, $h^{-1}\circ f\circ h$ is contained in $U_\varepsilon(\mathrm{id.})$. Put $U=U_\delta(\mathrm{id.})\cap U_\varepsilon(\mathrm{id.})$.

Let $\Psi_{\mathrm{id.}}: U \longrightarrow P^h(\mathrm{Diff}_0^c(L))$ be a map defined by

$$\psi_{id.}(f)(t)=\begin{cases}(1-2t)s_f & \text{for } 0\leq t\leq \frac{1}{2},\\ (2t-1)s_{h^{-1}\circ fh} & \text{for } \frac{1}{2}\leq t\leq 1,\end{cases}$$

where s_f in $\Gamma_c(T(L))$ corresponds to f in U by a coordinate mapping. $\psi_{id.}$ is a continuous map and $q\circ\psi_{id.}(f)=f$. Hence $\psi_{id.}$ is a local section. Let $U_\varepsilon(f)$ be a neighborhood of f in $\mathrm{Diff}_0^c(L)$ which is homeomorphic to the set $\{s\in\Gamma_c(f^*T(L));\ \|s\|<\varepsilon\}$, and $U_\varepsilon(h^{-1}\circ fh)$ be a neighborhood of $h^{-1}\circ fh$ in $\mathrm{Diff}_0^c(L)$. Let ℓ_t be a smooth path connecting f and $h^{-1}\circ fh$. Let U be a small neighborhood of f such that $U\subset U_\varepsilon(f)$, and for any f in U, $h^{-1}\circ fh$ is contained in $U_\varepsilon(h^{-1}\circ fh)$. Let $\psi_f: U\to P^h(\mathrm{Diff}_0^c(L))$ be a map defined by

$$\psi_f(f')(t)=\begin{cases}(1-3t)s_{f'}, & \text{for } 0\leq t\leq 1/3,\\ \ell_{3t-1} & \text{for } 1/3\leq t\leq 2/3,\\ (3t-2)s_{h^{-1}\circ f'h} & \text{for } 2/3\leq t\leq 1.\end{cases}$$

ψ_f is continuous and $q\circ\psi_f(f')=f'$. Hence ψ_f is a local section.

Q.E.D.

The fiber of the fibration q is the space of based loops in $Diff_0^c(L)$, which is denoted by $\Omega(Diff_0^c(L))$. Consider the homotopy exact sequence of the fibration q. Then we have

Proposition 3.5. $\pi_i(P^h(Diff_0^c(L))=0$ for each $i \geq 0$.

Corollary 3.6. $P^h(Diff_0^c(L))$ is a connected component of $P^h(Diff^c(L))$.

§4. Theorems

Note that the kernel of $\rho_1 \oplus \dots \oplus \rho_\lambda$ in Lemma 2.3 is the space $P^{h_1}(Diff^c(L_1)) \times \dots \times P^{h_\lambda}(Diff^c(L_\lambda))$. Hence each connected component of $\mathcal{J}$ is contractible from Proposition 3.5. Consider the homotopy exact sequence of the fibration **res** in Lemma 2.2,

$$\dots \to \pi_2(\mathcal{J}) \to \pi_2(\mathcal{E}) \to \pi_2(Diff_0(V_1) \times \dots \times Diff_0(V_\lambda)) \to \pi_1(\mathcal{J}) \to$$
$$\pi_1(\mathcal{E}) \to \pi_1(Diff_0(V_1) \times \dots \times Diff_0(V_\lambda)) \xrightarrow{\Delta} \pi_0(\mathcal{J}) \to \dots .$$

Let s be the number of V_i homeomorphic to a torus T^2. By the result of Earle and Eells[2], Gramain[5], $\mathrm{Diff}_0(T^2)$ is homotopy equivalent to T^2, and the other group $\mathrm{Diff}_0(V)$ is contractible. (Note that V_i is not diffeomorphic to S^2.) Thus the map $\Delta : \pi_1(\mathrm{Diff}_0(V_1) \times \ldots \times \mathrm{Diff}_0(V_\lambda)) \to \pi_0(\mathcal{G})$ reduces to the map $\Delta : (Z \oplus Z) \oplus \overset{s}{\ldots} \oplus (Z \oplus Z) \to \pi_0(\mathcal{G})$. By considering the holonomy around T^2, we may assume that $\Delta|_{(Z\oplus 0)\oplus \overset{s}{\ldots} \oplus (Z\oplus 0)}$ is injective. Therefore combining Lemma 2.1, we have

Theorem 4.1. $\pi_i(\mathrm{LDiff}(M,F)) = \begin{cases} 0 & \text{for } i \geqq 2, \\ \overset{\ell}{\oplus} Z & \text{for } i=1,\ 0 \leqq \ell \leqq s. \end{cases}$

Theorem 4.2. Let (M,F) be a generalized Reeb foliated 3-dimensional manifold. Then $\mathrm{FDiff}_0(M,F)$ has the same homotopy type as an ℓ-dimensional torus $T^\ell (0 \leqq \ell \leqq r+s)$, where r is the number of generalized Reeb components and s is the number of compact leaves homeomorphic to T^2.

Proof. Consider the homotopy exact sequence of the fibration in Lemma 1.13,

$$\cdots \to \pi_2(\mathrm{LDiff}(M,F)) \to \pi_2(\mathrm{FDiff}_0(M,F)) \to \pi_2(S^1 \overbrace{\times \cdots \times}^{r} S^1) \to$$

$$\pi_1(\mathrm{LDiff}(M,F)) \to \pi_1(\mathrm{FDiff}_0(M,F)) \to \pi_1(S^1 \overbrace{\times \cdots \times}^{r} S^1) \to \pi_0(\mathrm{LDiff}(M,F))$$

$$\to 1.$$

Since $\mathrm{FDiff}_0(M,F)$ is a topological group, $\pi_1(\mathrm{FDiff}_0(M,F))$ is an abelian group. Therefore we have

$$\pi_i(\mathrm{FDiff}_0(M,F)) = \begin{cases} 0 & \text{for } i \geqq 2, \\ \overset{\ell}{\oplus} Z & \text{for } i=1,\ 0 \leqq \ell \leqq r+s. \end{cases}$$

Hence $\mathrm{FDiff}_0(M,F)$ is weak homotopy equivalent to an ℓ-dimensional torus T^ℓ. By a result of Palais[8], $\mathrm{FDiff}_0(M,F)$ is homotopy equivalent to T^ℓ for $0 \leqq \ell \leqq r+s$. Q.E.D.

Let F a codimension one foliation on $S^1 \times S^2$ such that $F|_{S^1 \times D_i^2}(i=1,2)$ is a Reeb component, where $S^1 \times S^2 = S^1 \times D_1^2 \cup_{id.} S^1 \times D_2^2$.

Example 4.3. $\mathrm{FDiff}_0(S^1 \times S^2, F)$ is homotopy equivalent to $S^1 \times S^1$.

Proof. First we consider about the homotopy type of LDiff $(S^1 \times S^2, F)$, which is the fiber of the fibration π in Lemma 1.13. From the contractibility of $\mathrm{Diff}(D^2; \partial D^2)$, we see that $\mathcal{G}$ is homotopy equivalent to $Z \oplus Z$(see Lemma 2.3). In this case, ρ is an epimorphism.) Consider the homotopy exact sequence of the fibration res in Lemma 2.2,

$$\cdots \to \pi_1(\mathcal{G}) \to \pi_1(\mathcal{E}) \to \pi_1(\mathrm{Diff}_0(T^2)) \xrightarrow{\Delta} \pi_0(\mathcal{G}) \to \pi_0(\mathcal{E}) \to 1.$$

From the structure of the foliation F, $\Delta|_{Z \oplus 0}$ is an injection. Combining Lemma 2.1, we have

$$\pi_i(\mathrm{LDiff}(S^1 \times S^2, F); \mathrm{id.}) = \begin{cases} 0 & \text{for } i \geq 2, \\ Z & \text{for } i = 1, 2. \end{cases}$$

Next, consider the homotopy exact sequence of the fibration π in Lemma 1.13,

$$\cdots \to \pi_2(S^1 \times S^1) \to \pi_1(\mathrm{LDiff}(S^1 \times S^2, F)) \to \pi_1(\mathrm{FDiff}_0(S^1 \times S^2, F))$$

$$\to \pi_1(S^1 \times S^1) \to \pi_0(\mathrm{LDiff}(S^1 \times S^2, F)) \to 1.$$

Hence we have $\pi_i(\mathrm{FDiff}_0(S^1 \times S^2, F), \mathrm{id.}) = \begin{cases} Z \oplus Z & \text{for } i=1, \\ 0 & \text{for } i \geq 2. \end{cases}$ Q.E.D.

§5. Foliation induced from spinnable structures

A compact 3-dimensional manifold M is called spinnable if there exists a 1-dimensional submanifold X, which is a finite union of circle's, called an axis, satisfying the following conditions

(1) The normal bundle of X is trivial,

(2) Let $X \times D^2$ be a tubular neighborhood of X, then $M - X \times \mathrm{Int}\, D^2$ is the toal space of a fibration ξ over a circle S^1, and

(3) Let $p: M - X \times \mathrm{Int}\, D^2 \longrightarrow S^1$ be the projection of ξ, then the diagram

$$\begin{array}{ccc} X \times S^1 & \xrightarrow{\iota} & M - X \times \mathrm{Int}\, D^2 \\ & {\scriptstyle p'} \searrow & \downarrow {\scriptstyle p} \\ & & S^1 \end{array}$$

commutes, where ι denotes the inclusion map and p' denotes the projection onto the second factor.

The fiber L of ξ is called a generator and the pair $\mathcal{S} = (X, \xi)$

is called a spinnable structure on M.

Theorem 5.1 (Alexander[1]). Every closed orientable 3-dimensional manifold has a spinnable structure.

Let $(M,\mathcal{S})$ be a closed manifold with a spinnable structure $\mathcal{S}$. The axis X is a finite union of circle's, i.e., $\bigcup_{i=1}^{r} S^1_i$.

Hence we can construct a codimension one foliation on M from this spinnable structure(see[12]). We denote this foliation by $F_{\mathcal{S}}$. Note that $F_{\mathcal{S}}$ is a generalized Reeb foliation. Thus by Theorem 4.2, we know that $FDiff_0(M,F_{\mathcal{S}})$ has the same homotopy type of $\overbrace{S^1\times\dots\times S^1}^{\ell}$ for some integer ℓ , $0\leqq\ell\leqq 2r+1$. In this case we obtain a better information.

Theorem 5.2. $FDiff_0(M,F_{\mathcal{S}})$ is homotopy equivalent to T^{ℓ} for some ℓ , $0\leqq\ell\leqq r+1$, where r is equal to the number of connected components of the axis of $\mathcal{S}$.

Theorem 5.3. $\pi_i(LDiff(M,F_{\mathcal{S}});id.)=0$ for $i\geqq 1$, and

$\pi_0(\mathrm{LDiff}(M,F_{\mathcal{S}});\mathrm{id.})=0$ for $r=1$, $g=1$, where g is the genus of the generator L of $\mathcal{S}$.

Theorem 5.2 is proved from Theorem 5.3 using the same method as in the proof of Theorem 4.2. Moreover we have the following corollary of Theorems 5.2 and 5.3, which is a result of Fukui-Ushiki[4].

Corollary 5.4. $\mathrm{FDiff}_0(M,F_{\mathcal{S}})$ is homotopy equivalent to $S^1 \times S^1$ for $r=1$, $g=0$.

Proof of Theorem 5.3. By putting $V_i=T_i^2(\lambda=r)$ in §2, we have Lemmas 2.1,2.2 and 2.3. Consider the homotopy exact sequence of the fibration res in Lemma 2.2,

$$\cdots\to\pi_2(\mathrm{Diff}_0(T_1^2)\times\cdots\times\mathrm{Diff}_0(T_r^2))\to\pi_1(\mathcal{G})\to\pi_1(\mathcal{E})\to\pi_1(\mathrm{Diff}_0(T_1^2)\times$$

$$\cdots\times\mathrm{Diff}_0(T_r^2))\longrightarrow\pi_0(\mathcal{G})\to\pi_0(\mathcal{E})\longrightarrow 1.$$

From the structure of the foliation F around each compact leaf T_i^2, we see that Δ is injective. Hence we complete the proof of the first part of Theorem 5.3.

Next, we consider the case $r=1$, $g=0$. The kernel of ρ, in Lemma 2.3, is the space $P^h(\mathrm{Diff}(D^2;\partial D^2)) \times P^{id.}(\mathrm{Diff}(D^2;\partial D^2))$. Since $\mathrm{Diff}(D^2;\partial D^2)$ is contractible[10], $P^h(\mathrm{Diff}(D^2;\partial D^2)) \times P^{id.}(\mathrm{Diff}(D^2;\partial D^2))$ is connected. Q.E.D.

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