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A remark on the continuous variation of secondary characteristic classes for foliations

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§1. Introduction.

Thurston [3] has constructed a family of codimension n foliations on certain $(2n+1)$ manifold with continuously varying Godbillon-Vey invariant establishing a surjection,

$$gv : H_{2n+1}(B\Gamma_n ; \mathbb{Z}) \longrightarrow \mathbb{R} \longrightarrow 0.$$

The purpose of this note is to show that, using his results, we can show that some secondary characteristic classes other than that of Godbillon-Vey vary also continuously. We can also show that these classes are independent. Thus we can construct a surjection

$$H_{2n+1}(B\Gamma_n ; \mathbb{Z}) \longrightarrow \mathbb{R}^{r(n)+1} \longrightarrow 0$$

where $r(n)$ is a number depending on n. (See §3).

The foliations which we are going to construct to realize some characteristic classes are product foliations. The reason why these classes vary continuously is that they are in some sense "decomposable." The Godbillon-Vey class is "indecomposable" in the sense that it vanishes on any product foliation. In §3, we shall remark that there are infinitely many series of indecomposable classes, first of which are those of Godbillon-Vey.

To evaluate the characteristic classes on product foliations, we need the "Cartan formula" for the secondary classes. This will be done

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in §2.

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§2. The Cartan formula.

Suppose we are given two foliations $(M, \mathcal{F})$ and $(N, \mathcal{G})$ of codimensions p and q respectively. Then we can construct the product foliation $(M \times N, \mathcal{F} \times \mathcal{G})$. The classifying map for this foliation factors through $B\Gamma_p \times B\Gamma_q$;

$$M \times N \longrightarrow B\Gamma_p \times B\Gamma_q \xrightarrow{\mu} B\Gamma_n$$

where $n = p + q$ and μ is induced from the natural inclusion $\Gamma_p \times \Gamma_q \subset \Gamma_{p+q}$.

To calculate the characteristic classes of the product foliation $(M \times N, \mathcal{F} \times \mathcal{G})$, it is necessary to identify the image of the classes in $H^*(B\Gamma_n ; \mathbb{R})$ under the homomorphism μ^*. To do this, let us define characteristic classes for codimension n foliations in a slightly different way from that of Bott [2]*. Recall that Bott used the class c_i, which is the degree $2i$ part of $\det(I - \frac{1}{2\pi}\Omega)$ where Ω is the curvature matrix. We simply replace c_i by $\Sigma_i = (\frac{-1}{2\pi})^i \operatorname{Tr} \Omega^i$ and let γ_{2i+1} be the form corresponding to h_{2i+1} in the Bott's definition. Let WO'_n be a differential graded algebra defined by

* This idea is due to Lazarov.

$$WO_n' = \hat{\mathbb{R}}\ [\Sigma_1, \ldots, \Sigma_n] \otimes E(\gamma_1, \gamma_3, \ldots, \gamma_{2k+1})$$

$$\deg \Sigma_i = 2i, \quad \deg \gamma_{2i+1} = 4i + 1, \quad \text{and}$$

$$d\,\Sigma_i = 0, \quad d\,\gamma_{2i+1} = \Sigma_{2i+1}\ .$$

Then by exactly the same way as in [2], we obtain a homomorphism

$$H^*(WO_n') \longrightarrow H^*(B\Gamma_n\ ;\ \mathbb{R}).$$

Now let us define a homomorphism of differential graded algebras

$$\alpha : WO_n' \longrightarrow WO_p' \otimes WO_q'$$

by

$$\alpha(\Sigma_i) = \Sigma_i \otimes 1 + 1 \otimes \Sigma_i$$

$$\alpha(\gamma_{2i+1}) = \gamma_{2i+1} \otimes 1 + 1 \otimes \gamma_{2i+1}.$$

Then we have

Proposition 1 (Cartan formula)

The following diagram is commutative.

$$\begin{array}{ccc}
H^*(B\Gamma_n\ ;\ \mathbb{R}) & \xrightarrow{\ \mu^*\ } & H^*(B\Gamma_p \times B\Gamma_q\ ;\ \mathbb{R}) \\
 & & \uparrow \ \text{cross product} \\
\uparrow & & H^*(B\Gamma_p\ ;\ \mathbb{R}) \otimes H^*(B\Gamma_q\ ;\ \mathbb{R}) \\
 & & \uparrow \\
H^*(WO_n') & \xrightarrow{\ \alpha_*\ } & H^*(WO_p') \otimes H^*(WO_q')
\end{array}$$

Proof. Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two foliations of codimension p and q respectively on a smooth manifold M. Assume that $\mathcal{F}_1$ and $\mathcal{F}_2$ are transversal. Then $\mathcal{F} = \mathcal{F}_1 \cap \mathcal{F}_2$ is a codimension n $(= p + q)$ foliation on M. The normal bundle $\nu(\mathcal{F})$ is canonically isomorphic to

$\nu(\mathcal{F}_1) \oplus \nu(\mathcal{F}_2)$. Since the interior direct sum of Riemannian (resp. Bott) connections on $\nu(\mathcal{F}_1)$ and $\nu(\mathcal{F}_2)$ give rise to those of $\nu(\mathcal{F})$ and since we are using the class Σ_i, which is, up to a scalar, just $\operatorname{Tr} \Omega^i$, our Proposition is obvious for this particular case. Then the general case follows from the usual argument (cf [2]).

§3. Main theorem.

Let S be a complex analytic surface constructed by Kodaira (cf [1]), having the following properties.

(i) S is the total space of a fibre bundle over a curve with fibre another curve.

(ii) sign $(S) \neq 0$.

Let $\xi \subset T(S)$ be the tangent bundle along the fibres. From (ii) we conclude that

$$P_1(\xi) = 3 \text{ sign } S \neq 0.$$

According to Thurston [5], $B\overline{\Gamma}_2$ is 3-connected. Therefore again by Thurston [4], ξ is homotopic to the normal bundle of a codimension 2 foliation $\mathcal{F}$. We have

$$P_1(\nu(\mathcal{F})) = 3 \text{ sign } S \neq 0.$$

Now let $(M^{2n+1}, \mathcal{G}^n_t)$ be the family of codimension n foliations on a manifold M^{2n+1} constructed by Thurston [3], such that

$$< gv(\mathcal{G}^n t) \ , \ [M^{2n+1}] > = t \in \mathbb{R}.$$

where t ranges over some open set of $\mathbb{R}$. We consider the product foliation

$$\underbrace{(S,\mathcal{F}) \times \ldots \times (S,\mathcal{F})}_{i} \times (M^{2n+1-4i},\mathcal{G}_t).$$

These are a family of codimension n foliations on $(2n + 1)$ manifold $(S)^i \times M^{2n+1-4i}$. We claim that some characteristic class in $H^{2n+1}(B\Gamma_n ; \mathbb{R})$ varies continuously on this family.

More precisely we have the following. Let $r(n)$ be the greatest integer satisfying the inequality $2n + 1 - 4r(n) \geq 3$. Then we have

Theorem. There is a surjection

$$H_{2n+1}(B\Gamma_n ; \mathbb{Z}) \longrightarrow \mathbb{R}^{r(n)+1} \longrightarrow 0$$

for any $n \geq 1$.

Proof. We consider the following family of foliations.

$$(N^i,\mathcal{H}^i_t) = ((S)^i \times M^{2n+1-4i},\mathcal{F}^i \times \mathcal{G}_t) \qquad i = 0,1,..,r(n).$$

Next we consider characteristic classes $\gamma_1\Sigma_1^n$, $\gamma_1\Sigma_1^{n-2}\Sigma_2$, ..., $\gamma_1\Sigma_1^{n-2r(n)}\Sigma_2^{r(n)}$.

We claim that

(1) $\gamma_1\Sigma_1^{n-2i}\Sigma_2^i\,(\mathcal{H}^i{}_t) = i\,!\,\cdot(6 \text{ sign } S)^i \cdot t \qquad i = 0,\ldots,r(n).$

(2) $\gamma_1\Sigma_1^{n-2i}\Sigma_2^i\,(\mathcal{H}^j{}_t) = 0 \qquad \text{for } j > i.$

Clearly our theorem follows from these two statements.

Now we first verify (1). We have a map

$$\alpha : WO'_n \longrightarrow WO'_2 \otimes \ldots \otimes WO'_2 \otimes WO'_{n-2i},$$

which is an iteration of the maps of type $WO'_n \longrightarrow WO'_p \otimes WO'_q$ considered in §2. Let $x \in H^*(WO'_2 \otimes \ldots \otimes WO'_2 \otimes WO'_{n-2i})$ be a cohomology class, and let

$(a_1, a_2, \ldots, a_{i+1})$ be an $(i+1)$-tuple of non-negative integers. We define $\mathcal{X}_{(a_1, a_2, \ldots, a_{i+1})}$ to be the multi-degree $(a_1, a_2, \ldots, a_{i+1})$ part of $\mathcal{X}$. Then we have, by Proposition 1,

$$\alpha_*([\gamma_1 \Sigma_1^{n-2i} \Sigma_2^i])_{(4,4,\ldots,4,2n+1-4i)}$$

$$= i\,!\; [\Sigma_2] \otimes [\Sigma_2] \otimes \ldots \otimes [\Sigma_2] \otimes [\gamma_1 \Sigma_1^{n-2i}].$$

Therefore

$$< [\gamma_1 \Sigma_1^{n-2i} \Sigma_2^i]\,(\varphi^i t),\quad [N^i] > = i\,!\,(6 \text{ sign } S)^i \cdot t \quad .$$

Next we prove (2).

The same calculation as above using proposition 1 yields

$$\alpha_*([\gamma_1 \Sigma_1^{n-2i} \Sigma_2^i])_{(4,4,\ldots,4,2n+1-4j)} = 0 \qquad \text{for} \quad j > i.$$

This proves (2) and hence our Theorem.

Remark. Let us call an element $x \in H^{2n+1}(WO_n')$ "indecomposable" if $\alpha_*(x) = 0$ for any factorization $\alpha : WO_n' \longrightarrow WO_{i_1}' \otimes \ldots \otimes WO_{ik}' (i_1 + \ldots + i_k = n)$.

Let x be such an element. Then obviously,

$$x\,(\text{product foliation}) = 0 \quad .$$

It is easy to show that the classes

$$\gamma_{2i+1} \Sigma_{j_1}^{k_1} \ldots \Sigma_{j_\ell}^{k_\ell} \in H^{2n+4i+1}(WO_{n+2i}')$$

$$(n = j_1 k_1 + \ldots + j_\ell k_\ell)$$

are indecomposable if j_s is odd for every $s = 1, \ldots, \ell$. Thus we have infinitely many series of indecomposable elements. For example,

$$\gamma_1\Sigma_1 \ , \ \gamma_1\Sigma_1^2 \ , \ \gamma_1\Sigma_1^3 \ , \ \ldots$$

$$\gamma_1\Sigma_3 \ , \ \gamma_1\Sigma_1\Sigma_3 \ , \ \gamma_1\Sigma_1^2\Sigma_3 \ , \ \ldots$$

$$\gamma_1\Sigma_5 \ , \ \gamma_1\Sigma_1\Sigma_5 \ , \ \gamma_1\Sigma_1^2\Sigma_5 \ , \ \ldots$$

(In Bott's definition, γ_1 corresponds to h_1, Σ_1 corresponds to c_1, Σ_2 corresponds to $c_1^2 - 2c_2$ and so on). We also have another type of indecomposable elements, e.g. $\gamma_1\Sigma_2 \in H^5(WO_2')$.

By the argument in this note, the problem of continuous variation of characteristic classes in $H^{2n+1}(WO_n')$ splits into the following two problems.

(1) To construct a family of codimension n foliation on some M^{2n+1}'s on which indecomposable elements take values continuously and independently.

(2) To show that the natural map $\pi^* : H^{4k}(BGL_{2k}\mathbb{R}\,;\mathbb{R}) \to H^{4k}(B\Gamma_{2k}\,;\mathbb{R})$ is injective. (This is the case for $k = 1$ by Thurston [5].)

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