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Growth of a Spherical Nucleus in the Undercooling State of Metal

by

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1. Introduction and Formulation **

It is well known that when a pure liquid is cooled to its melting point T_E, the solid phase may form; alternatively if nucleation is suppressed, the liquid will continue to cool (that is, undercool or supercool).

The free energy change on forming solid from liquid at the equilibrium transformation temperature T_E is

$$\Delta G = \Delta H - T_E \Delta S = 0$$

where ΔG, ΔH and ΔS are molar changes in free energy, enthalpy and entropy, respectively. At temperature T different from T_E,

$$\Delta G = \Delta H - T\Delta S \neq 0.$$

Neglecting the small temperature-dependence of ΔH and ΔS and combining last two equations yields

$$\Delta G = \frac{\Delta H \, \Delta T}{T_E}$$

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**) About the physical base we rely on [1].

where ΔT is the undercooling ($= T_E - T$), whose value is negative for solidification.

An important influence on the melting point of a pure material is surface curvature. The surface curvature can be viewed as introducing an excess pressure in the solid phase (only). The excess pressure is accounted as follows :

$$\Delta p = 2\,\sigma\kappa$$

where σ is surface free energy per unit area and κ is the mean surface curvature. If solid cannot be compressed, $\Delta p = \Delta G$ holds. Thus

$$\frac{\Delta H\,\Delta T}{T_E} = 2\,\sigma\kappa$$

or

$$\frac{1}{\kappa} = \frac{2\sigma T_E}{\Delta H \Delta T} = \frac{2\sigma T_E}{L(T_E - T_I)} \quad ,$$

where L is latent heat and T_I is surface temperature.

A crystal smaller than a critical size is called "embryo", which grows to nucleus with the critical size. We are concerned about growth of the nucleus. For simplicity we consider spherical nucleus, whose size is determined just by its radius. The critical radius is

$$r^* = \frac{2\sigma T_E}{L(T_E - T_A)} \quad ,$$

T_A is initial liquid temperature of undercool state.

From the above stated we can describe a system of equations which determines the radius Y(t) and the temperature distribution T(r,t) as follows :

$$\begin{cases} \rho c \dfrac{\partial T}{\partial t} = k\Delta T \ , \quad 0 < r < Y(t) \ , \quad t > 0 \ , \\ -k \dfrac{\partial T}{\partial r} (Y(t),t) = L\rho \dot{Y}(t) \ , \quad t > 0 \ , \\ Y(0) = \dfrac{2\sigma T_E}{L(T_E - T_A)} \\ \dfrac{L(T_E - T_I)}{2\sigma T_E} = \dfrac{1}{Y(t)} \ , \quad t > 0 \quad (T_I = T(Y(t),t)) \ , \\ T(r,0) = T_0(r) \ , \end{cases}$$

where k is thermal conductivity, ρ density, c specific heat of solid metal, and L latent heat, which are assmed to be constant. Putting

$$\frac{T - T_A}{T_E} = u \ , \quad \frac{kT_E}{L\rho} = \alpha \ , \quad \frac{2\sigma}{L} = m \ .$$

$$\frac{2\sigma T_E}{L(T_E - T_A)} = b \ , \quad Y(t) = y(t) \text{ and } T_0(r) = f(r)$$

yields the simplified form ·

$$(P) \begin{cases} (P\text{-}1) & \dfrac{\partial u}{\partial t} = \Delta u \ , \quad 0 < r < y(t) \ , \quad t > 0 \ , \\ (P\text{-}2) & u(y(t),t) = \dfrac{m}{b} - \dfrac{m}{y(t)} \ , \quad t > 0 \ , \\ (P\text{-}3) & \dot{y}(t) = -\alpha \dfrac{\partial u}{\partial r} (y(t),t) \ , \quad t > 0 \ , \\ (P\text{-}4) & y(0) = b \ , \\ (P\text{-}5) & u(r,0) = f(r) \ , \quad 0 \le r \le b \ . \end{cases}$$

Here we shall refer to the series of Friedman's works, Part I (1959) [2], Part II (1960)[3] and Part III (1960)[4]. In Part I he considered the problem of melting of solids. In Part II he considered the development of one liquid drop surrounded by totally supersaturated or totally undersaturated vapour of its own substance. Denoting by u(r,t) the normalized vapour density at the point (r,t), we get for u(r,t) the following system of equations

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u , \quad y(t) < r < \infty , \quad t > 0 , \\ u(y(t),t) = 1 \text{ (saturation density)}, \quad t > 0 , \\ \dot{y}(t) = \alpha \dfrac{\partial u}{\partial r} (y(t),t), \quad t > 0 , \\ y(0) = b \ (b > 0), \\ u(r,0) = f(r), \quad r > b . \end{cases}$$

In Part III he considered the behavior of one gas bubble in liquid in which some of same gas is dissolved. Denoting by u(r,t) the normalized density of dissolved gas in the liquid, we get for u(r,t) the following system of equations

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u , \quad y(t) < r < \infty , \quad t > 0 , \\ u(y(t),t) = \dfrac{m}{y(t)} - \dfrac{m}{b} , \quad t > 0 , \\ (1 + \dfrac{n}{y(t)})\dot{y}(t) = \alpha \dfrac{\partial u}{\partial r} (y(t),t), \quad t > 0 , \\ y(0) = b , \\ u(r,0) = f(r), \quad r > b . \end{cases}$$

We note that the case of m=0, n=0 reduces to that of Part II by putting 1-u=v. He could prove the unique existence of the solution only for small α.

Our problem is analogous to Friedman's problems, especially that of Part III. Thus it can be viewed as that of Part IV. Indeed we can prove unique existence of the solution analogously. By putting

$$r = x \ , \quad ru(r,t) = w(x,t) \ , \quad rf(r) = \phi(x)$$

our system takes the form

$$(P')\quad \begin{cases} \dfrac{\partial w}{\partial t} = \dfrac{\partial^2 w}{\partial x^2} \ , \quad 0 < x < y(t), \quad t > 0 \ , \\ w(y(t),t) = \dfrac{m}{b}\,(y(t) - b), \quad t > 0 \ , \\ w(0,t) = 0 \ , \quad t > 0 \ , \\ y(t)\dot{y}(t) = -\,\alpha \dfrac{\partial w}{\partial x}\,(y(t),t) + \alpha\left(\dfrac{m}{b} - \dfrac{m}{y(t)}\right), \quad t > 0 \ , \\ y(0) = b \ , \\ w(x,0) = \phi(x) \ , \quad 0 < x < b \ . \end{cases}$$

By introducing the Green's functions of heat equation

$$g(x,t;\ \xi,\tau) = \frac{1}{2\sqrt{\pi(t-\tau)}}\, e^{-\frac{(x-\xi)^2}{4(t-\tau)}} - \frac{1}{2\sqrt{\pi(t-\tau)}}\, e^{-\frac{(x+\xi)^2}{4(t-\tau)}}$$

$$G(x,t;\ \xi,\tau) = \frac{1}{2\sqrt{\pi(t-\tau)}}\, e^{-\frac{(x-\xi)^2}{4(t-\tau)}} + \frac{1}{2\sqrt{\pi(t-\tau)}}\, e^{-\frac{(x+\xi)^2}{4(t-\tau)}}$$

and making use of standard technique, we get the nonlinear integral equation of volterra type for $u_x(y(t),t)$ equivalent to the last system as followings :

$$
(P'')\begin{cases} v(t) = -2\int_0^b G_\xi(y(t),t;\ \xi,0)\phi(\xi)d\xi + \frac{2m}{b}\int_0^t \dot{y}(\tau)G(y(t),t;\ y(\tau),\tau)d\tau \\ \qquad -2\int_0^t G_\xi(y(t),t;\ y(\tau),\tau)v(\tau)d\tau \\ y\dot{y} + \alpha(\frac{m}{y} - \frac{m}{b}) + \alpha v(t) = 0 \end{cases}
$$

where $v(t) = \lim_{x\to y(t)} u_x(x,t)$. If the last system for $v(t)$ and $\dot{y}(t)$ is solved, $w(x,t)$ (and hence $u(x,t)$) is given by the formula

$$
w(x,t) = \int_0^b g(x,t;\ \xi,0)\phi(\xi)d\xi + \frac{m}{b}\int_0^t g(x,t;\ y(\tau),\tau)(y(\tau) - b)\dot{y}(\tau)d\tau
$$
$$
+ \int_0^t \{g(x,t;\ y(\tau),\tau)v(\tau) - \frac{m}{b}\ u(y(\tau),\tau)\frac{\partial g}{\partial \xi}(x,t;\ y(\tau),\tau)\}d\tau
$$

The system (P'') can be solved by Picard's iteration procedure for small t and small α. Using monotonicity of $\dot{y}(t)$ (that is itself to be maked sure of) allows to continue the solution for all t.

In the method, however, restiction on α is very severe. Here we take another more constructive way. This way is one taken by Cannon, Hill and Primicerio for the most simple Stefan's problem [5], [6], [7].

Here we will give our definition of solution :

Definition *A pair of functions (y,u) is called a solution of the system (P) in $0 \le t \le T$, if the following conditions are satisfied :*

i) $y(t) \in C(0 \le t \le T)$, $\dot{y}(t) \in C(0 < t \le T)$,

ii) $u(r,t)$, $\frac{\partial u}{\partial r}(r,t) \in C(0 \le r \le y(t),\ 0 < t \le T)$,

$\Delta u \in C(0 \le r < y(t),\ 0 < t \le T)$, *iii)* (y,u) *satisfies (P-1)* $\sim$ *(P-5).*

In item 2 we will give some lemmata necessary for later purposes. In item 3 uniqueness and existence of solution of our problem will be proved. In item 5 we will propose a difference scheme solving our problem and give another proof of existence theorem as a by-product. In item 6 iteration procedure for solving our difference scheme will be given and its convergen will be proved.

2. Some lemmata

Here we shall collect some a priori properties of solution (y,u) of the system (P).

First we assume that

$$(2.1)\quad \begin{cases} f(r) \text{ is continuous,} \\ f(r) \geq 0\ ,\quad f(r) \not\equiv 0\ ,\quad 0 \leq r \leq b\ . \end{cases}$$

Lemma 1 *Under the assumption (2.1),* $y(t)$ *is strictly monotone increasing for* $t > 0$, *that is*

$$\dot{y}(t) > 0 \quad (t > 0).$$

(Proof) Assume $y(t) < 0$ in an interval $0 \leq t \leq t_0$, then the function $U \equiv u - \left(\frac{m}{b} - \frac{m}{y(t)} \right)$ satisfies the system of equations

$$\begin{cases} L_0 U \equiv \Delta U - U_t = \dfrac{m\dot{y}(t)}{y(t)^2} < 0\ , \\ U(y(t),t) = 0\ , \\ U(r,0) = f(r)\ . \end{cases}$$

By the maximum principle we have $U \geq 0$ $(0 \leq t \leq t_0)$, and hence $\frac{\partial U}{\partial r}(y(t),t) = \frac{\partial u}{\partial r}(y(t),t) \leq 0$. This means $\dot{y}(t) \geq 0$ through (P-3), which is, however, contradiction to our assumption $\dot{y}(t) < 0$. Thus from continuity of $\dot{y}(t)$ we get $\dot{y}(t) \geq 0$ for some interval $0 \leq t \leq t_0$, and

$$u(r,t) \geq 0 \quad (0 \leq r \leq y(t) \ , \ 0 \leq t \leq t_0)$$

Next we assume that

$$\dot{y}(t_0) = 0 \ , \quad \dot{y}(t) < 0 \quad (t_0 < t \leq t_1).$$

Then the function $U = u - (\frac{m}{y(t_0)} - \frac{m}{y(t)})$ satisfies

$$\begin{cases} L_0 U = \Delta U - U_t = \dfrac{m\dot{y}(t)}{y(t)^2} < 0 \quad (t_0 < t \leq t_1) \ , \\ U(y(t),t) = \dfrac{m}{b} - \dfrac{m}{y(t_0)} > 0 \ , \\ U(r,t_0) = u(r,t_0) \geq 0 \ . \end{cases}$$

By the maximum principle we have $U \geq 0$ $(t_0 \leq t \leq t_1)$ and arrive at contradiction analogously as done above. Thus we get $\dot{y}(t) \geq 0$ for all $t \geq 0$.

Next we shall show that the equality can be omitted. If the equality were to happen, there would be t' and t'' such that

$$y(t') \equiv y(t) \ , \quad t' \leq t \leq t'' \quad (\dot{y}(t) = 0, \quad t' \leq t \leq t'')$$

Since above defined U satisfies $L_0 U = 0$ and $U(y(t),t) = 0$ $(t' \leq t \leq t'')$. From $f(r) \geq 0$ and $f(r) \not\equiv 0$, by the strong maximum principle we get $U > 0$ and by Friedman's lemma [8], $\frac{\partial U}{\partial r} = \frac{\partial u}{\partial r}(y(t),t) < 0$, which is contradiction to $\dot{y}=0$ $(t' \leq t \leq t'')$. Thus the proof is completed.

Next we assume that

(2.2) $f(r) \le N(\frac{1}{r} - \frac{1}{b})$ (N : positive constant).

<u>*Lemma 2*</u> *Under the assumption (2.1) and (2.2)*

$$(2.3) \qquad 0 < \frac{\dot{y}(t)}{\alpha} = - u_r(y(t),t) \le \frac{N}{y(t_0)^2} \qquad (0 \le t \le t_0)$$

holds for any t_0.

(Proof) Introduce a 'barrier' function

$$\omega_{t_0}(r,t) = N[\frac{1}{r} - \frac{1}{y(t_0)}] + \frac{m}{b} - \frac{m}{y(t_0)} .$$

Since, by Lemma 1

$$\omega_{t_0}(y(t),t) = N[\frac{1}{y(t)} - \frac{1}{y(t_0)}] + \frac{m}{b} - \frac{m}{y(t_0)} > \frac{m}{b} - \frac{m}{y(t)} \quad (0 \le t < t_0),$$

$$\omega_{t_0}(r,0) = N[\frac{1}{r} - \frac{1}{y(t_0)}] + \frac{m}{b} - \frac{m}{y(t_0)} > N[\frac{1}{r} - \frac{1}{y(t_0)}]$$

$$> N(\frac{1}{r} - \frac{1}{b}) \ge f ,$$

we have $\omega_{t_0} - u \ge 0$ $(0 \le t \le t_0)$ and hence

$$\frac{u(y(t_0) - \rho, t_0) - (\frac{m}{b} - \frac{m}{y(t_0)})}{\rho} \le \frac{N}{(y(t_0) - \rho)y(t_0)} \quad (\rho > 0),$$

which produces

$$- \frac{\partial u}{\partial r}(y(t_0),t_0) \le \frac{N}{y(t_0)^2}$$

Since t_0 is arbitrary, we get the right side inequality of (2.3). The left side inequality is also true from Lemma 1.

Further we suppose that

(2.4) $f'(r) \le 0$.

Lemma 3 *Under the condition (2.1) and (2.4)*

$$\frac{\partial u}{\partial r}(r,t) \le 0 \quad \text{for} \quad 0 \le r \le y(t) \quad \text{and} \quad t > 0$$

holds.

(Proof) $z = \frac{\partial u}{\partial r}$ satisfies the system of equations

$$\begin{cases} \Delta z - \frac{2}{r^2} z - \frac{\partial z}{\partial t} = 0 , & t > 0 , \\ z(y(t),t) = -\frac{1}{\alpha} \dot{y}(t) < 0 , & \\ z(r,0) = f'(r) \le 0 , & \\ z(0,t) = 0 . & \end{cases}$$

Suppose that there exist an interior point $(\bar{r},\bar{t})$ such that

$$\max_{\substack{0 \le r \le y(t) \\ 0 \le t \le t_0}} z(r,t) = z(\bar{r},\bar{t}) = M > 0 , \quad z(r,t) < M (t < \bar{t})$$

Then at $(\bar{r},\bar{t})$, $z_r = 0$, $z_{rr} \le 0$ and $z_t \ge 0$ hold, but this means that $\Delta z - \frac{2}{r^2} z - z_t < 0$, which is contradiction. Therefore we get $z(r,t) \le 0$.

Next we shall state a comparison theorem. We consider two pairs of data (b_i,f_i) $(i=1,2)$ satisfying conditions (2.1) and (2.4), and their corresponding solutions (y_i,u_i) $(i=1,2)$.

Lemma 4 *If $b_1 < b_2$ and $f_1 \le f_2$, then*

$$y_1(t) < y_2(t) \quad (t \ge 0)$$

holds.

(Proof) If the statement is not true, there is a t_0 such that

$$y_1(t_0) = y_2(t_0), \quad \dot{y}_1(t_0) \ge \dot{y}_2(t_0), \quad y_1(t) < y_2(t) \quad (0 \le t < t_0)$$

In $0 < t < t_0$, we have by Lemma 3

$$u_2(y_1(t),t) \geq \frac{m}{b} - \frac{m}{y_2(t)} > \frac{m}{b} - \frac{m}{y_1(t)} = u_1(y_1(t),t)$$

and hence by the maximum principle

$$u_2(r,t) - u_1(r,t) > 0 \quad (0 < r < y_1(t),\ 0 < t < t_0).$$

Since $u_2(y_1(t_0),t_0) - u_1(y_1(t_0),t_0) = 0$, it follows by Friedman's lemma that

$$\frac{\partial u_2}{\partial r}(y_2(t_0),t_0) - \frac{\partial u_1}{\partial r}(y_1(t_0),t_0) < 0\ , \quad \text{i.e.} \quad \dot{y}_2(t_0) > \dot{y}_1(t_0),$$

which is contradiction. Thus we have completed the proof.

Here we will give a fundamental formula. Multiplying (P-1) by r^2 and integration of the result over the region $\{0 < r < y(\tau),\ 0 < \tau < t\}$ yields

$$\int_0^{y(t)} u(r,t)r^2dr - \int_0^b u(r,0)r^2dr - \int_0^t u(y(\tau),\tau)y^2(\tau)\dot{y}(\tau)d\tau$$
$$= \int_0^t y^2(\tau)\ \frac{\partial u}{\partial r}\ (y(\tau),\tau)d\tau\ ,$$

which can be written by using the conditions (P-2) $\sim$ (P-5) as

$$\int_0^{y(t)} u(r,t)r^2dr - \int_0^b f(r)r^2dr - \int_0^t \left(\frac{m}{b} - \frac{m}{y(\tau)}\right)y^2(\tau)\dot{y}(\tau)d\tau$$
$$= -\frac{1}{\alpha}\int_0^t y^2(\tau)\dot{y}(\tau)d\tau\ .$$

From this by integration we get the desired fundamental formula

$$\frac{1}{3}\left(\frac{1}{\alpha} - \frac{m}{b}\right)(y^3 - b^3) + \frac{m}{2}(y^2 - b^2)$$
$$= \int_0^b f(r)r^2dr - \int_0^{y(t)} u(r,t)r^2dr\ , \tag{2.5}$$

which is useful for later purposes.

From the last formula we have first a comparison theorem containing the case of $b_1 = b_2$ (see Lemma 4).

Lemma 4' *If $b_1 \le b_2$ and $f_1 \le f_2$, then*

$$y_1 \le y_2 \quad (t \ge 0)$$

holds.

(Proof) It is sufficient to prove it for the case of $b_1 = b_2$, which we assume. Take an arbitrary positive constant δ, and consider three pairs of data (b_1, f_1), (b_2, f_2) and $(b_2 + \delta, f_2)$. (The last f_2 is taken to be zero for $b_2 < r \le b_2 + \delta$.) It is clear from the previous lemma that the corresponding solutions (y_1, u_1), (y_2, u_2) and (y_2^δ, u_2^δ) satisfy

$$(2.6) \qquad y_1 < y_2^\delta \quad , \quad y_2 < y_2^\delta \qquad (t \ge 0).$$

Applying the fundamental formula (2.5) to two solutions (y_2^δ, u_2^δ) and (y_2, u_2), we have

$$\frac{1}{3}\left(\frac{1}{\alpha} - \frac{m}{b_2+\delta}\right)(y_2^\delta)^3 + \frac{m}{2}(y_2^\delta)^2$$
$$= \frac{1}{3}\left(\frac{1}{\alpha} - \frac{m}{b_2+\delta}\right)(b_2+\delta)^3 + \frac{m}{2}(b_2+\delta)^2 + \int_0^{b_2+\delta} f_2(r)r^2dr - \int_0^{y_2^\delta(t)} u_2^\delta(r,t)r^2dr$$

and

$$\frac{1}{3}\left(\frac{1}{\alpha} - \frac{m}{b_2}\right)y_2^3 + \frac{m}{2}y_2^2$$
$$= \frac{1}{3}\left(\frac{1}{\alpha} - \frac{m}{b_2}\right)b_2^3 + \frac{m}{2}b_2^2 + \int_0^{b_2} f_2(r)r^2dr - \int_0^{y_2(t)} u_2(r,t)r^2dr .$$

Subtracting the latter equation from the former yields

$$\frac{1}{3}\left(\frac{1}{\alpha}-\frac{m}{b_2}\right)[(y_2^{\delta})^3 - y_2^3] + \frac{m}{2}[(y_2^{\delta})^2 - y_2^2]$$

(2.7)

$$= \frac{1}{3}\left(\frac{m}{b_2+\delta}-\frac{m}{b_2}\right)(y_2^{\delta})^3 + \frac{1}{3}\left(\frac{1}{\alpha}-\frac{m}{b_2}\right)[(b_2+\delta)^3 - b_2^3]$$
$$+ \frac{1}{3}\left(\frac{m}{b_2}-\frac{m}{b_2+\delta}\right)(b_2+\delta)^3 + \frac{m}{2}[(b_2+\delta)^2 - b_2^2]$$
$$- \int_0^{y_2(t)} [u_2^{\delta}(r,t) - u_2(r,t)]r^2 dr - \int_{y_2(t)}^{y_2^{\delta}(t)} u_2^{\delta}(r,t)r^2 dr .$$

By the maximum principle we see that

$$u_2^{\delta}(r,t) \geq 0 \quad (0 \leq r \leq y_2^{\delta}(t)) \quad \text{and}$$

$$u_2^{\delta}(r,t) - u_2(r,t) \geq 0 \quad (0 \leq r \leq y_2(t)).$$

Hence we have from (2.7)

(2.8)

$$\frac{1}{3}\left(\frac{1}{\alpha}-\frac{m}{b_2}\right)(y_2^{\delta})^3 + \frac{m}{2}(y_2^{\delta})^2$$
$$\leq \frac{1}{3}\left(\frac{1}{\alpha}-\frac{m}{b_2}\right)y_2^3 + \frac{m}{2}y_2^2 - \frac{\delta m}{3b_2(b_2+\delta)}(y_2^{\delta})^3$$
$$+ \frac{1}{3}\left(\frac{1}{\alpha}-\frac{m}{b_2}\right)[(b_2+\delta)^3 - b_2^3] - \frac{\delta m}{3b_2(b_2+\delta)}(b_2+\delta)^3$$
$$+ \frac{m}{2}[(b_2+\delta)^2 - b_2^2].$$

We note that the function

$$F(y) = \frac{1}{3}\left(\frac{1}{\alpha}-\frac{m}{b}\right)y^3 + \frac{m}{2}y^2$$

is monotone increasing for all $y > 0$ if $b > \alpha m$, and also for $0 < y < \frac{\alpha m b}{\alpha m - b}$ if $b < \alpha m$.

Now it follows from continuity and monotonicity (Lemma 1) of $y_2^{\delta}(t)$

and $y_1(t)$ that for sufficiently small t and δ,

$$b_2 + \delta \le y_2^{\delta}(t) \quad (\le \frac{\alpha m b_2}{\alpha m - b_2} \ , \ \text{if } b_2 < \alpha m \text{) and}$$

$$b_1 = b_2 \le y_1(t) \quad (\le \frac{\alpha m b_2}{\alpha m - b_2} \quad \text{if } b_2 < \alpha m \text{).}$$

Therefore by the above note and hence we get from (2.6) and (2.8)

$$\frac{1}{3}\left(\frac{1}{\alpha} - \frac{m}{b_2}\right)y_1^{\ 3} + \frac{m}{2}y_1^{\ 2} \le \frac{1}{3}\left(\frac{1}{\alpha} - \frac{m}{b_2}\right)y_2^{\ 3} + \frac{m}{2}y_2^{\ 2} + O(\delta)$$

for sufficiently small t and δ. The last term $O(\delta)$ tends to zero as $\delta \to 0$ since $y_\delta(t)$ is uniformly bounded for small t and δ. Since δ is arbitrary, the inequality implies

$$\frac{1}{3}\left(\frac{1}{\alpha} - \frac{m}{b_2}\right)y_1^{\ 3} + \frac{m}{2}y_1^{\ 2} \le \frac{1}{3}\left(\frac{1}{\alpha} - \frac{m}{b_2}\right)y_2^{\ 3} + \frac{m}{2}y_2^{\ 2}$$

for sufficiently small t. Again by the above note we get

(2.9) $\quad y_1(t) \le y_2(t)$ for sufficiently small t.

If $f_1 \equiv f_2$, $b_1 = b_2$, we have also $y_1(t) \ge y_2(t)$, and hence $y_1(t) = y_2(t)$ for sufficiently small t. Continuing the same reasoning we get $y_1(t) = y_2(t)$ for all $t > 0$.

Suppose that $f_1 \not\equiv f_2$, $f_1 \le f_2$, $b_1 = b_2$. If the equality $y_1(t) = y_2(t)$ holds for an interval $0 \le t \le t_0$, the maximum principle yields $u_1(r,t) - u_2(r,t) < 0$ $(0 < r < y_1(t),\ 0 < t \le t_0)$. Hence by Friedman's lemma

$$\frac{\partial u_1}{\partial r}(y_1(t),t) - \frac{\partial u_2}{\partial r}(y_1(t),t) > 0 \quad \text{i.e.} \quad \dot{y}_1(t) < \dot{y}_2(t)$$

holds for $0 \le t \le t_0$, which is contradiction to our assumption. Therefore we get $y_1(t) < y_2(t)$ and $u_1(r,t) < u_2(r,t)$ for sufficiently small t, hence by applying Lemma 4

$$y_1(t) < y_2(t) \quad \text{for all} \quad t > 0 .$$

Thus we have proved Lemma 4'.

We will investigate the behavior of $y(t)$ as $t \to \infty$.

<u>Lemma 5</u> *Suppose that there exists a solution (y,u) of (P) for all $t > 0$. Then*

$$\lim_{t \to \infty} y(t) = y(\infty)$$

where $y(\infty)$ is the positive root of the equation (2.10)

(Proof) Consider the solution $w(r,t)$ of the system

$$\begin{cases} w_t - \Delta w = 0 & (r > 0,\ t > 0) \\ w(r,0) = \begin{cases} f(r) - (\frac{m}{b} - \frac{m}{r}) & 0 < r < b \\ 0 \le b < r \end{cases} \end{cases}$$

It is clear that $u(r,t) - (\frac{m}{b} - \frac{m}{r})$ is bounded above by $w(r,t)$.

However we have as $t \to \infty$

$$r^2 w(r,t) = r \int_0^b g(r,t;\ \xi,0)\{f(\xi) - (\frac{m}{b} - \frac{m}{\xi})\}\xi d\xi$$

$$< \frac{1}{2\sqrt{\pi}\, t^{3/2}} r^2 e^{-\frac{(r-b)^2}{4t}} \int_0^b \{f(\xi) - (\frac{m}{b} - \frac{m}{\xi})\}\xi^2 d\xi$$

$$< \frac{b}{2\sqrt{\pi}\, t^{3/2}} \left(\frac{b+\sqrt{b^2+16t}}{2}\right)^2 \max_{0 \le r \le b} r^2 \{f(r) - (\frac{m}{b} - \frac{m}{r})\}$$

Thus $\lim_{t \to \infty} r^2 w(r,t) = 0$ uniformly, and hence $\lim_{t \to \infty} r^2\{u(r,t) - (\frac{m}{b} - \frac{m}{r})\} = 0$

Therefore from (2.5) we have

$$\frac{1}{3\alpha}\, y(\infty)^3 = \frac{1}{3\alpha}\, b^3 + \frac{m}{6}\, b^2 + \int_0^b f(r) r^2 dr \tag{2.10}$$

Hence we get a unique root $y(\infty)$ of the last equation, and by the monotonicity

$$y(t) < y(\infty) \quad (t \geq 0),$$

$$\lim_{t \to \infty} y(t) = y(\infty).$$

3. Uniqueness and existence of solution

Directly from Lemma 4' we have a uniqueness theorem.

Theorem 1 *The system (P) has not more than one solution under the condition (2.1) and (2.4).*

(Proof) Suppose that there are two pairs of solution (y_1, u_1) and (y_2, u_2). Then $y_1 \equiv y_2$ holds from Lemma 4'. Hence the well known theory of the problem with prescribed boundary shows $u_1 \equiv u_2$.

Next we will construct a solution of our problem . For that we consider an auxiliary problem, which we call (AP), with a prescribed boundary $r = y_0(t)$:

$$\text{(AP)} \begin{cases} \dfrac{\partial u}{\partial t} = \Delta u\ , \quad 0 < r < y_0(t)\ , \quad t > t_0\ , \\ u(y(t),t) = \dfrac{m}{b} - \dfrac{m}{y_0(t)}\ , \quad t > t_0\ , \\ u(r,t_0) = f_0(r)\ , \quad 0 \leq r \leq y_0(t_0) \end{cases}$$

Putting

$$t_0 = 0\ ,$$

$$f_0(r) = f^\theta(r) = \begin{cases} f(r) , & 0 \le r \le b - \theta \\ 0 , & b - \theta \le r \le b \end{cases}$$

$$y_0(t) = y^\theta(t) = b , \quad 0 < t < \theta ,$$

we get the corresponding solution u^θ of (AP), and we know that $\frac{\partial u^\theta}{\partial r}$ exists and is continuous for $0 \le r \le y^\theta(t)$, $0 < t \le \theta$ and that

$$(3.1) \qquad 0 \le - \frac{\partial u^\theta}{\partial r} \le \frac{N}{(y^\theta(t))^2}$$

under the asumption (2.2).

Suppose that by the boundary function

$$(3.2) \qquad y^\theta(t) = b - \alpha \int_0^t \frac{\partial u^\theta}{\partial r} (y^\theta(\tau - \theta), \tau - \theta) d\tau \qquad (t \ge \theta)$$

the solution u^θ of (AP) is constructed recursively for $\theta \le t \le n\theta$, and $\frac{\partial u^\theta}{\partial r}$ exists and is continuous, and further $0 \le - \frac{\partial u^\theta}{\partial r} (y^\theta(t),t) \le \frac{N}{y^\theta(t-\theta)^2}$. Then $y^\theta(t)$ is continuously differentiable for $t \ge \theta$. Now in $n\theta \le t \le (n+1)\theta$, we define $y^\theta(t)$ by (3.2) and solve (AP) with $t_0 = n\theta$, $y_0(t) = y^\theta(t)$, $f_0(r) = u^\theta(r,n\theta)$, and then get $u^\theta(r,t)$. From the above assumption $y^\theta(t)$ turns out to be continuously differentiable also for $n\theta \le t \le (n+1)\theta$ and

$$(3.3) \qquad 0 \le \dot{y}^\theta(t) \le \frac{\alpha N}{\{y^\theta(t - \theta)\}^2} .$$

Continuing the procedure we get $\{y^\theta, u^\theta\}$ for $0 \le t \le T$ and we know that (3.3) holds for $\theta \le t \le T$.

The family of functions $\{y^\theta\}$ is equicontinuous and uniformly bounded in $0 \le t \le T$. By Ascoli-Arzela's theorem, hence, we can extract a subsequence of $\{y^\theta\}$ which converges to a Lipshitz continuous function $y(t)$. Define $u(r,t)$ as the solution of (AP) with $t_0 = 0$, $f_0(r) = f(r)$ and $y_0(t) = y(t)$ (defined above). We can also prove by the maximum principle that the corresponding subsequence $\{u_\theta(r,t)\}$ converges to $u(r,t)$.

So defined (y,u) turns out to be the desired solution of our problem (P). In fact, since (y,u) satisfies (P-1), (P-2), (P-4) and (P-5), it remains to assure (P-3), which is, as easily been shown, equivalent to the fundamental formula (2.5) under Lipshitz continuity of $y(t)$. For (y^θ,u^θ) we have an analogous formula

$$\int_0^t (y^\theta)^2\left(-\frac{1}{\alpha}\dot{y}(\tau+\theta)\right)d\tau - \int_0^{y(t)} u^\theta(r,t)r^2dr + \int_0^b f^\theta(r)r^2dr$$
$$+ \int_0^t \left(\frac{m}{b} - \frac{m}{y^\theta(\tau)}\right)(y^\theta)^2\dot{y}^\theta d\tau = 0$$

Tending θ to zero through the extracted subsequence, we get the fundamental formula (2.5) for the limit functions (y,u). Hence (y,u) satisfies also (P-3). It is clear from uniqueness that all sequence (y^θ,u^θ) converges to (y,u). Thus we have proved existence of solution under the assumption (2.2) and that it can be obtained by the above stated method.

We can get rid of the condition (2.2), and we have

<u>*Theorem 2*</u> *Suppose that f(r) is continuously differentiable, not equal to zero identically and satisfies (2.1)(2.4)*. Then there is a pair of solution (y,u) of our problem (P).*

---

*) This implies that $f(b)\neq 0$ may occur. Then the solution u is considered at (b,0) to satisfy $0 \le \varliminf_{(r,t)\to(b,0)} u(r,t) \le \varlimsup_{(r,t)\to(b,0)} u(r,t) \le f(b)$

Indeed define

$$f^{\delta}(r) = \begin{cases} f(r), & 0 \le r < b - \delta \ , \\ 0 \ , & b - \delta \le r \le b \end{cases}$$

for each δ satisfying $0 < \delta < b$. Since f is bounded in $0 \le r \le b$, it follows that for each $\delta(0 < \delta < b)$ there exists an $N = N(\delta)$ such that

$$0 \le f^{\delta}(r) \le N(\delta)\left(\frac{1}{r} - \frac{1}{b} \right).$$

Hence, for each δ there exists a unique solution (y^{δ}, u^{δ}) of the problem (P) corresponding to the data (b, f^{δ}). From Lemma 4' we see that for $\delta_2 < \delta_1$, $y^{\delta_1} \le y^{\delta_2}$. Set $\delta_n = 2^{-n}b$, $n=1,2, \ldots$, and consider $\{(y^{\delta_n}, u^{\delta_n})\}$. Now the sequence $\{y^{\delta_n}\}$ is a monotone increasing sequence of functions which are bounded above via Lemma 4' by the free boundary of the solution of the problem (P) corresponding to the data $(b+1, f)$ where f is extended as zero over $b < x \le b+1$. Consequently, there exists a function $y(t) = \lim_{n \to \infty} y^{\delta_n}(t)$. From Lemma 1 y^{δ_n}, $n=1,2, \ldots$, are strictly monotone increasing functions of t. Hence, for $\sigma > 0$, $y^{\delta_1}(\sigma) - b > 0$ and it follows from the boundedness of f that there exists a positive constant $K = K(f, y^{\delta_1}(\sigma) - b)$ such that for each δ_n and t_0 satisfying $0 < \sigma \le t_0 \le T$ we have by Lemma 2

$$0 < \frac{\dot{y}^{\delta_n}(t_0)}{\alpha} = - u_r(y^{\delta_n}(t_0), t_0) \le \frac{K}{y^{\delta_n}(t_0)^2} .$$

Since t_0 is arbitrary, we get for all $n=1,2, \ldots$

$$0 < \dot{y}^{\delta_n}(t) \le \frac{\alpha K}{y^{\delta_n}(t)^2} \le \frac{\alpha K}{b^2}$$

for $\sigma \le t \le T$. Thus it follows that the limit function y is Lipshitz continuous with Lipshitz constant $\frac{\alpha K}{b^2}$ for $\sigma \le t \le T$. In order to demonstrate the continuity of y at t=0, consider the solution (ρ,v) of the problem

$$(3.4)\quad \begin{cases} \frac{\partial v}{\partial t} - \Delta v = 0 \ , \quad b < r < \rho(t) \ , \quad 0 < t \le T \ , \\ v(b,t) = M \ , \quad 0 < t \le T \ , \\ v(\rho(t),t) = \frac{m}{b} - \frac{m}{\rho(t)} \ , \quad 0 < t \le T \ , \\ \dot{\rho}(t) = -\alpha \frac{\partial v}{\partial r}(\rho(t),t) \ , \quad 0 < t \le T \ , \\ \rho(0) = b \ . \end{cases}$$

where $M = \max\left(\max_{0 \le r \le b} f(r), \frac{m}{b} \right)$.

It can be seen that its solution exists and $\rho(t)$ is monotone increasing function for all $t \ge 0$ (see item 4), and further that $y^{\delta_n}(t) \le \rho(t)$ holds for all $t > 0$ by the same reasoning as in the proofs of Lemma 4 and Lemma 4'. Therefore we get $y(t) \le \rho(t)$ for all $t > 0$. Since $y^{\delta_1}(t) \le y(t) \le \rho(t)$ and both $y^{\delta_1}(t)$ and $\rho(t)$ are continuous for $t \ge 0$ ($y^{\delta_1}(0) = \rho(0) = b$), it follows that y(t) is also continuous at t=0. Let u denote the solution of (P-1),(P-2),(P-5) with boundary y(t). It follows from the maximum principle that $u^{\delta_n} \to u$ as $n \to \infty$. Hence (y,u) satisfies the fundamental formula (2.5) which implies that y is continuously differentiable for $t > 0$ and that (y,u) is the solution of (P) for the data (b,f).

4. Comment on unique solvability of the problem (3.4)

First we note that the following system

$$
(4.1)\quad \begin{cases}
\dfrac{\partial v^\delta}{\partial t} - \Delta v^\delta = 0 \;, \quad b < r < \rho^\delta(t) \;, \quad 0 < t \le T \;, \\[2mm]
v^\delta(b,t) = M \;, \quad 0 < t \le T \;, \\[2mm]
v^\delta(\rho^\delta(t),t) = \dfrac{m}{b+\delta} - \dfrac{m}{\rho^\delta(t)} \;, \quad 0 < t \le T \;, \\[2mm]
\dot{\rho}^\delta(t) = -\alpha \dfrac{\partial v^\delta}{\partial r}(\rho^\delta(t),t) \;, \quad 0 < t \le T \;, \\[2mm]
\rho^\delta(0) = b + \delta \;, \\[2mm]
v^\delta(r,0) = \dfrac{m}{b+\delta} - \dfrac{m}{r} \quad (b < r \le b+\delta)
\end{cases}
$$

has the unique solution (ρ^δ, v^δ). (Here $0 \le \lim\limits_{(r,t)\to(b,0)} v^\delta(r,t) \le$ $\le \lim\limits_{(r,t)\to(b,0)} v^\delta(r,t) \le M$.) In fact by repeating the almost same argument of Lemma 1 we get

$$
(4.2)\qquad \dot{\rho}_\delta(t) > 0 \quad (t > 0)
$$

By using a 'barrier' function

$$
\omega_{t_0}(r,t) = \frac{M}{\dfrac{1}{b} - \dfrac{1}{\rho^\delta(t_0)}} \left[\frac{1}{r} - \frac{1}{\rho^\delta(t_0)} \right] + \frac{m}{b+\delta} - \frac{m}{\rho^\delta(t_0)}
$$

we get through the analogous argument of Lemma 2.

$$
(4.3)\qquad 0 < \frac{\dot{\rho}_\delta(t)}{\alpha} = - v_r^\delta(\rho^\delta(t),t) \le \frac{Mb}{\rho^\delta(t_0)(\rho^\delta(t_0)-b)} \le \frac{M}{\rho^\delta(t_0)-b}
$$

Noting that $v_r^\delta(b,t) \le 0$ (by the maximum principle) and repeating the argument in Lemma 3 yields also

$$
(4.4)\qquad v_r^\delta(r,t) \le 0 \;, \; b \le r \le \rho^\delta(t) \;, \quad t > 0 \;.
$$

From this we get also the correspondings to Lemma 4, that is

(4.5) $\rho^{\delta_1}(t) < \rho^{\delta_2}(t) \quad (t \geq 0), \; (\delta_1 < \delta_2)$

And further in order to prove

(4.6) $\rho^{\delta_1}(t) = \rho^{\delta_2}(t) \quad (t \geq 0), \; (\delta_1 = \delta_2)$

we use another fundamental formula

(4.7)
$$\frac{1}{3b}\left(\frac{1}{\alpha} - \frac{m}{b+\delta}\right)\{\rho^{\delta}(t)^3 - (b+\delta)^3\} - \frac{1}{2}\{\frac{1}{\alpha} - (\frac{m}{b} + \frac{m}{b+\delta})\} \times$$
$$\times \{\rho^{\delta}(t)^2 - (b+\delta)^2\} - m\{\rho^{\delta}(t) - (b+\delta)\}$$
$$= m\int_0^t \frac{d\tau}{\rho^{\delta}(\tau)} - \int_b^{\rho(t)} r^2(\frac{1}{b} - \frac{1}{r})v^{\delta}(r,t)dr + (M - \frac{n}{b-\delta})t +$$
$$+ \int_b^{b+\delta} r^2(\frac{1}{b} - \frac{1}{r})(\frac{m}{b+\delta} - \frac{m}{r})dr$$

which is gotten by using the Green's formula

$$\oint r^2 uv^{\delta}dr + (r^2 u\frac{\partial v^{\delta}}{\partial r} - r^2 v^{\delta}\frac{\partial u}{\partial r})d\tau = 0$$

along the boundary of the region $b < r < \rho^{\delta}(t)$, $0 < \tau < t$ with $u = \frac{1}{b} - \frac{1}{r}$. Subtractig the formula with $\delta=\delta_1$ for that with $\delta=\delta_2$ $(\delta_1 < \delta_2)$ yields

$$\frac{1}{3b}\left(\frac{1}{\alpha} - \frac{m}{b+\delta_1}\right)\rho^{\delta_2}(t)^3 - \frac{1}{2}\{\frac{1}{\alpha} - (\frac{m}{b} + \frac{m}{b+\delta_1})\}\rho^{\delta_2}(t)^2 - m\rho^{\delta_2}(t)$$
$$= \frac{1}{3b}\left(\frac{1}{\alpha} - \frac{m}{b+\delta_1}\right)\rho^{\delta_1}(t)^3 - \frac{1}{2}\{\frac{1}{\alpha} - (\frac{m}{b} + \frac{m}{b+\delta_1})\}\rho^{\delta_1}(t) - m\rho^{\delta_1}(t)$$
$$- \int_b^{\rho^{\delta_1}(t)} (\frac{1}{b} - \frac{1}{r})(v^{\delta_2}(r,t) - v^{\delta_1}(r,t))r^2 dr$$
$$- \int_{\rho^{\delta_1}(t)}^{\rho^{\delta_2}(t)} (\frac{1}{b} - \frac{1}{r})v^{\delta_2}(r,t)r^2 dr + m\int_0^t \left(\frac{1}{\rho^{\delta_2}(\tau)} - \frac{1}{\rho^{\delta_1}(\tau)}\right)d\tau + O(\delta_2 - \delta_1)$$

By noting that $v^{\delta_2}(r,t) - v^{\delta_1}(r,t) > 0$ $(0 < r < \rho^{\delta_1}(t))$, $v^{\delta_2}(r,t) > 0$ and $\rho^{\delta_2}(t) > \rho^{\delta_1}(t)$ we get

$$\frac{1}{3b}\left(\frac{1}{\alpha}-\frac{m}{b+\delta_1}\right)\rho^{\delta_2}(t)^3 - \frac{1}{2}\left\{\frac{1}{\alpha}-\left(\frac{m}{b}+\frac{m}{b+\delta_1}\right)\right\}\rho^{\delta_2}(t)^2 - m\rho^{\delta_2}(t)$$

$$\le \frac{1}{3b}\left(\frac{1}{\alpha}-\frac{m}{b+\delta_1}\right)\rho^{\delta_1}(t)^3 - \frac{1}{2}\left\{\frac{1}{\alpha}-\left(\frac{m}{b}+\frac{m}{b+\delta_1}\right)\right\}\rho^{\delta_1}(t)^2 - m\rho^{\delta_1}(t) + O(\delta_2-\delta_1).$$

Since the function $F(\rho) = \frac{1}{3b}\left(\frac{1}{\alpha}-\frac{m}{b+\delta_1}\right)\rho^3 - \frac{1}{2}\left\{\frac{1}{\alpha}-\left(\frac{m}{b}+\frac{m}{b+\delta_1}\right)\right\}\rho^2 - m\rho$ is monotone increasing for $\rho > b$ if $\rho - b$ is small, we get together with (4.5)

$$\rho^{\delta_1}(t) < \rho^{\delta_2}(t) \le \rho^{\delta_1}(t) + O(\delta_2 - \delta_1) \quad \text{for small } t.$$

Tending δ_2 to δ_1 we have (4.6) for small t, and hence for all t.

By noting (4.2) $\sim$ (4.6) we can construct a solution (ρ^δ, v^δ) in the same way as in item 3. This solution is uniquely determined by (4.6).

Next we shall show that there exist two positive constants β_1 and β_2 such that

$$\text{(4.8)} \qquad b + \beta_1\sqrt{t} < \rho^\delta(t) < b + \delta + \beta_2\sqrt{t} \quad (0 \le t \le \varepsilon).$$

Putting

$$r\left[v^\delta - \left(\frac{m}{b+\delta} - \frac{m}{r}\right)\right] = w^\delta$$

yields the system to be satisfied by w^δ :

$$\frac{\partial w}{\partial t} - \frac{\partial^2 w^\delta}{\partial r^2} = 0, \quad b < r < \rho^\delta(t), \quad 0 < t \le T,$$

$$w^\delta(b,t) = b\left[M + \frac{\delta}{b(b+\delta)}\right],$$

$$\begin{cases} w^\delta(\rho^\delta(t),t) = 0 , \\ \rho^\delta(t)\dot{\rho}^\delta(t) + \dfrac{\alpha\, m}{\rho^\delta(t)} = -\alpha \dfrac{\partial w^\delta}{\partial r}(\rho^\delta(t),t) , \quad 0 < t \le T , \\ \rho^\delta(0) = b + \delta , \\ w^\delta(r,0) = 0 . \end{cases}$$

For comparison let's introduce the function

$$(4.9) \qquad w(r,t) = \frac{2b}{\alpha} \sum_{n=1}^{\infty} \frac{1}{(2n)!} \frac{\partial^n}{\partial t^n} (r - y(t) - b)^{2n}$$

with a infinitely differentiable function y(t) for t>0. It is clear that w satisfies the following equations

$$(4.10) \begin{cases} \dfrac{\partial w}{\partial t} - \dfrac{\partial^2 w}{\partial r^2} = 0 , \\ w(y(t)+b,t) = 0 , \\ 2b\, \dot{y}(t) = -\alpha \dfrac{\partial w}{\partial r} (y(t)+b,t) . \end{cases}$$

If we take that $y_\beta(t) = \beta t^{\frac{1}{2}}$, we get

$$w(b,t) = w_\beta(b,t) = \frac{2b}{\alpha} \sum_{n=1}^{\infty} \frac{n!}{(2n)!} \beta^{2n}$$

and we know that

$$\frac{2b}{\alpha} (e^{\frac{1}{4}\beta^2} - 1) < w_\beta(b,t) < \frac{2b}{\alpha} (e^{\beta^2} - 1).$$

Hence putting $\beta_1 = [\log (1 + \frac{\alpha M}{2})]^{\frac{1}{2}}$ yields

$$w_{\beta_1}(b,t) < bM \,(< w^\delta(b,t)).$$

Then it follows that for sufficiently small t

$$(4.11) \qquad b + \beta_1 t^{\frac{1}{2}} < \rho^{\delta}(t)$$

holds. In fact, if that is not true, there exist a sufficiently small t_0 such that

$$b + \beta_1 t^{\frac{1}{2}} < \rho^{\delta}(t) \ (0 \le t < t_0), \quad b + \beta_1 t_0^{\frac{1}{2}} = \rho^{\delta}(t_0).$$

Then we have by the maximum principle

$$w^{\delta}(r,t) > w_{\beta_1}(r,t) \quad \text{for} \quad 0 < r < b+\beta_1 t^{\frac{1}{2}}, \quad 0 < t < t_0$$

and hence, since $w^{\delta}(\rho^{\delta}(t_0),t_0) = w_{\beta_1}(b+\beta_1 t_0^{\frac{1}{2}}, t_0) = 0$, by Friedman's theorem

$$\frac{\partial w^{\delta}}{\partial r}(\rho^{\delta}(t_0),t_0) < \frac{\partial w_{\beta_1}}{\partial r}(\rho^{\delta}(t_0),t_0).$$

This means that

$$\rho^{\delta}(t_0)\dot{\rho}^{\delta}(t_0) + \frac{\alpha\, m}{\rho^{\delta}(t_0)} > 2b\, \dot{y}(t_0).$$

By the assumption, $\dot{\rho}^{\delta}(t_0) \le \dot{y}(t_0)$ and hence

$$\frac{\alpha m}{b} > 2b\, \dot{y}(t_0) - \rho^{\delta}(t_0)\dot{\rho}^{\delta}(t_0)$$

$$> b\, \dot{y}(t_0) \quad \text{for sufficiently small } \delta \text{ and } t_0.$$

But this enequality can not occur for sufficiently small t_0 since $\dot{y}(t_0) = \dfrac{\beta_1}{2\sqrt{t_0}}$. Therefore the left inequality of (4.8) holds for sufficiently small t.

On the other hand we have from (4.3)

$$\dot{\rho}^{\delta}(t) \le \frac{\alpha\, M}{\rho^{\delta}(t) - b}$$

and hence

$$\frac{1}{2}\rho^{\delta}(t)^2 - b\rho^{\delta}(t) \le \frac{1}{2}(b+\delta)^2 - b(b+\delta) + \alpha M t$$

or

$$\rho^{\delta}(t) \le b + \sqrt{\delta^2 + 2\alpha M t}$$

$$< b + \delta + \beta_2\sqrt{t} \qquad (\beta_2 = \sqrt{2\alpha M})$$

Thus we have proved (4.8).

From (4.3) and (4.8) we get also

(4.12) $$0 < \dot{\rho}^{\delta}(t) < \frac{\alpha M}{\beta_1\sqrt{t}}$$

for $0 < t \le \varepsilon$.

Now we consider a sequence $\{(\rho^{\delta},v^{\delta})\}$ of solution as $\delta \to 0$. Taking $\delta < 1$ and putting $\sigma^{\delta}(t) \equiv t(\rho^{\delta}(t) - b)$, then we have from (4.8)

(4.13) $$0 < \sigma^{\delta}(t) < \varepsilon(1 + \beta_2\sqrt{\varepsilon}) \quad (0 \le t \le \varepsilon)$$

and from (4.8) and (4.12)

(4.14) $$0 < \dot{\sigma}^{\delta}(t) = \rho^{\delta}(t) - b + t\dot{\rho}^{\delta}(t) \le 1 + (\beta_2 + \frac{\alpha M}{\beta_1})\sqrt{\varepsilon}, \quad (0<t\le\varepsilon).$$

Thus $\{\sigma^{\delta}(t)\}$ are equicontinuous and uniformly bounded over the interval $[0,\varepsilon]$. Therefore there is a subsequence, which we denote by $\{\sigma_n(t)\}$ $(n \to \infty)$, such that $\sigma_n(t)$ converges uniformly to a Lipschitz continuous function $\sigma(t)$. Then we put $\rho_n(t) = t^{-1}\sigma_n(t)$ and $\rho(t) = t^{-1}\sigma(t)$. From (4.8) we get $b + \beta_1\sqrt{t} \le \rho(t) \le b + \beta_2\sqrt{t}$ and from (4.5) $\rho_{n+1}(t) < \rho_n(t)$.

We consider $v_n(r,t)$ corresponding to $\rho_n(t)$ and the solution $v(r,t)$

of the auxiliary problem (3.4)', that is, the problem with the prescribed boundaries $\rho(t)$ and the fourth condition of (3.4) being deleted. From (4.12) and (3.4)

$$0 \le v_n(\rho(t),t) - (\frac{m}{b} - \frac{m}{\rho_n(t)}) \le \frac{\alpha M}{\beta_1 \sqrt{t}} (\rho_n(t) - \rho(t))$$

$$= \frac{\alpha M}{\beta_1 t^{3/2}} (\sigma_n(t) - \sigma(t)).$$

From the continuity of the solutions $v(t)$ and $v_n(t)$, for any η there exist t_o and n_0 such that $\max_{b \le r \le \rho(t)} |v_n(r,t) - v(r,t)| < \eta$ for $0 \le t \le t_0$ and $n \ge n_0$. By the maximum principle

$$\max_{\substack{b \le r \le \rho(t) \\ t_0 < t \le \varepsilon}} |v_n(r,t) - v(r,t)| < \max(\eta, \frac{M}{\beta_1 t_0^{3/2}} \max_{t_0 < t \le \varepsilon} (\sigma n_0(t) - \sigma(t)))$$

Therefore we get $v_n(r,t) \to v(r,t)$ $(n \to \infty)$. Resulted $\{\rho,v\}$ turns to be the desired solution for $0 < t \le \varepsilon$ of (3.4). In fact if $n \to \infty$ $(\delta \to 0)$ along the extracted sequence in the formula (4.7) it follows that

$$\frac{1}{3b}(\frac{1}{\alpha} - \frac{m}{b})(\rho(t)^3 - b^3) - (\frac{\alpha}{2} - \frac{m}{b})(\rho(t)^2 - b^2) - m(\rho(t) - b)$$

$$= m \int_0^t \frac{d\tau}{\rho(\tau)} - \int_b^{\rho(t)} r^2(\frac{1}{b} - \frac{1}{r}) v(r,t)dr + (M - \frac{m}{b})t$$

which is equivalent to the fourth condition of (3.4). Noting that $0 \le v(r,\varepsilon) - (\frac{m}{b} - \frac{m}{\rho(\varepsilon)}) \le \frac{\alpha M}{\beta_1 \sqrt{\varepsilon}} (\rho(\varepsilon) - r)$ by (4.12), we get the solution (ρ,v) for also $[\varepsilon,T]$ in the same way as in the proof of existence of the solution for (4.1). Thus we have proved that there exists a solution (ρ,v) of (3.4) for all $t \in [0,T]$.

Uniqueness can be shown as in Theorem 2.

Theorem 3 *There exists one and only one solution (ρ,v) of the* *system (3.4) and $\rho(t)$ satisfies*

$$\beta_1\sqrt{t} < \rho(t) - b < \beta_2\sqrt{t} \quad (\beta_1,\ \beta_2 : \text{positive constants}).$$

5. A Difference Scheme

Here we will propose a difference scheme for solving our problem (P). We shall introduce a family of rectangular lattices on the (r,t) plane with space mesh h and time steps k_n $(n=1,2, \ldots)$. Let's vary h so that $(b + \frac{1}{2}h)/h = J$ runs through integers and find k_n's so that the free boundary crosses lattices just at each mesh point (r_{J+n}, t_n), where $r_j = (j - \frac{1}{2})h$ $(j=0,1,2, \ldots)$ and $t_n = \sum_{p=1}^{n} k_p$. With reference to given positive numbers h and k_n we introduce the devided differences

$$u_r(r_j,t_n) = \frac{1}{h}\,[\,u(r_{j+1},t_n) - u(r_j,t_n)\,],$$

$$u_{\bar{r}}(r_j,t_n) = \frac{1}{h}\,[\,u(r_j,t_n) - u(r_{j-1},t_n)\,],$$

$$u_{r\bar{r}}(r_j,t_n) = \frac{1}{h^2}\,[\,u(r_{j+1},t_n) - 2u(r_j,t_n) + u(r_{j-1},t_n)\,],$$

and $$u_{\bar{t}}(r_j,t_n) = \frac{1}{k_n}\,[\,u(r_j,t_n) - u(r_j,t_{n-1})\,].$$

A difference analogue to our problem (P) is

$$(5.1)\quad \begin{cases} \Delta_h u(r_j,t_n) \equiv u_{r\bar{r}}(r_j,t_n) + \dfrac{1}{r_j}\,(u_r(r_j,t_n) + u_{\bar{r}}(r_j,t_n)) = u_{\bar{t}}(r_j,t_n) \\ u(r_{J+n},t_n) = \dfrac{m}{b} - \dfrac{m}{y_n} \quad (y_n = y_{n-1} + h \text{ or } y_{n-1} - h \text{ or } y_{n-1}\,). \\ u(r_1,t_n) = u(r_0,t_n) \\ \dfrac{y_n - y_{n-1}}{k_n} = -\alpha v_n + \beta\dfrac{k_n}{\sqrt{h}} \quad (\beta \text{ is a positive constant}; \\ \qquad\qquad\qquad v_n = u_{\bar{r}}(r_{J+n},t_n)) \end{cases}$$

$$u(r_j,0) = f(r_j),$$

where the third equation corresponds to $\frac{\partial u}{\partial r}(0,t) = 0$, which is automatically satisfied by the spherical simetricity for the solution of the differential problem (P). The fourth equation corresponds to (P-3) and contain an artificial added term $\beta \frac{k_n}{\sqrt{h}}$, which makes sure of convergen of the difference scheme and convergence of its iteratively solving procedure. (see also [9])

The quantities to be determined are $\{k_n\}$ and $\{u(r_j,t_n)\}$, while they are (y,u) in the original problem.

First we shall prepare a maximum principle for our difference scheme. Suppose that the function $U(r_j,t_n)$ $(-\frac{1}{2}h \le r_j \le y_n,\ n=0,1,2,\ \dots\ N)$ satisfies

$$
(5.2)\quad \begin{cases}
L_{0h}U(r_j,t_n) \equiv \Delta_h U(r_j,t_n) - U_{\bar{t}}(r_j,t_n) \le 0 \\
\qquad\qquad (0 < r_j < y_n,\ n=1,2,\ \dots\ ), \\
U(y_n,t_n) = F_n \ge 0 \quad (n=1,2,\ \dots\ ), \\
U(r_1,t_n) = U(r_0,t_n) \quad (n=0,1,2,\ \dots\ ), \\
U(r_j,0) = f(r_j) \ge 0 \quad (j=1,2,\ \dots\ ,\ J).
\end{cases}
$$

Then we have

<u>*Lemma 6*</u> *$U(r_j,t_n) \ge 0$ holds for $0 < r_j \le y_n$ and $n=0,1,2,\ \dots$. If further $f(r) \not\equiv 0$, it follows that $U(r_j,t_n) > 0$ for $0 < r_j < y_n$, $n=1,2,\ \dots$.*

(Proof) Putting $(j-\frac{1}{2})U(r_j,t_n) = V(r_j,t_n)$ reduces (5.2) to the following :

$$V_{r\bar{r}}(r_j,t_n) - V_{\bar{t}}(r_j,t_n) \le 0 \quad (0 < r_j < y_n,\ n=1,2,\ \dots\ ),$$

$$V(y_n,t_n) = \left(\frac{y_n + \frac{1}{2}h}{h} - \frac{1}{2}\right)F_n \quad (n=1,2,\ \dots\ ),$$

(5.3)

$$V(r_1,t_n) + V(r_0,t_n) = 0 \quad (n=0,1,2,\ \dots\ ),$$

$$V(r_j,0) = (j - \frac{1}{2})f(r_j) \quad (j=1,2,\ \dots\ ,\ J).$$

If there exists a point (r_{j_0},t_{n_0}) $(j_0 \ge 1)$ such that $V(r_j,t_n) \ge 0$ for $0 < r_j \le y_n$, $n=0,1,\ \dots\ ,\ n_0-1$ and $V(r_{j_0},t_{n_0}) < 0$ and further $V(r_j,t_{n_0}) \ge V(r_{j_0},t_{n_0})$ for $r_j \ne r_{j_0}$, $r_j > 0$, then it follows that

$$V_{r\bar{r}}(r_{j_0},t_{n_0}) \ge 0 \quad \text{and} \quad V_{\bar{t}}(r_{j_0},t_{n_0}) < 0\ ,$$

which is contradiction to the first equation of (5.3). Therefore we have $V(r_j,t_n) \ge 0$ and hence $U(r_j,t_n) \ge 0$ for $0 < r_j \le y_n$, $n=0,1,2,\ \dots\ .$

Next we suppose that $f(r) \ge 0$ and $f(r) \not\equiv 0$. If there is a point $r = r_{j_0}$ (> 0) on $t = t_1$ such that $V(r_{j_0},t_1) = 0$, then it follows the first equation of (5.3) that $V_{r\bar{r}}(r_{j_0},t_1) \le 0$ and hence by the fact $V \ge 0$, $V(r_{j_0\pm1},t_1) = 0$. Repeating this argument yields $V(r_j,t_1) = 0$ for all $r_j \le y_1$. But at a point r_{j_1} such that $f(r_{j_1}) > 0$, $V_{\bar{t}}(r_{j_1},t_1) < 0$ and $V_{r\bar{r}}(r_j,t_1) = 0$ hold, and that is contradiction to the first equation of (5.3). Thus we get $V(r_j,t_1) > 0$ for $0 < r_j < y_1$. Successively continuing the same reasoning step by step we get $V(r_j,t_n) > 0$ for all $0 < r_j < y_n$, $n=1,2,\ \dots\ ,$ and hence $U(r_j,t_n) > 0$ for all $0 < r_j < y_n$, $n=1,2,\ \dots\ .$ (Q.E.D)

For simplicity of notation we denote $u(r_j,t_n)$ by u_j^n occasionally.

Now we shall see the property of strictly monotone increasing of

$\{y_n\}$.

<u>*Lemma 7*</u> *Assume that (2.1)* $(f(r) \geq 0, f(r) \not\equiv 0)$ *holds. Then the solution* (y,u) *of (5.1) satisfies for* $n=1,2,3, \ldots$

$$y_n > y_{n-1}, \quad u_{\bar{r}}(y_n, t_n) < 0$$

and $0 < u_j^n \leq \max\left(\frac{m}{b}, \max_{0\leq r\leq b} f(r)\right)$

(Proof) (This is analogous to that of Lemma 1.) Assume that $y_{n_0} \leq y_{n_0-1} \leq \cdots \leq y_0 = b$. Then the function $U_j^n \equiv u_j^n - \left(\frac{m}{b} - \frac{m}{y_n}\right)$ satisfies

$$(5.4)\quad \begin{cases} L_{0h}U_j^n = \Delta_h U_j^n - (U_j)_{\bar{t}}^{\,n} = -\left(\dfrac{m}{y_n} - \dfrac{m}{y_{n-1}}\right)\dfrac{1}{\Delta t} \quad (\leq 0), \\ \qquad\qquad (j=1,2, \ldots J-n-1, \quad n=1,2, \ldots), \\ U_{J-n}^n = 0 \quad (n=1,2, \ldots), \\ U_1^n = U_0^n \quad (n=0,1,2, \ldots), \\ U_j^0 = f(r_j) \quad (j=1,2, \ldots). \end{cases}$$

By Lemma 6 we have $U_j^n > 0$ $(j=0,1, \ldots, J-n;\ n=1,2, \ldots)$ and hence $U_{\bar{r}}(y_n,t_n) = u_{\bar{r}}(y_n,t_n) < 0$. This implies that $y_n - y_{n-1} > 0$, which is contradiction to our assumption. Hence since n_0 is arbitrary, it follows that $y_n > y_{n-1}$ $(n=1,2, \ldots)$ and hence $u_{\bar{r}}(y_n,t_n) < 0$ $(n=1,2, \ldots)$.

Next we put

$$\max\left(\frac{m}{b}, \max_{0\leq r\leq b} f(r)\right) = M$$

and consider the function $V_j^n = M - u_j^n$ satisfing (5.2) with $V_{J+n}^n = M - (\frac{m}{b} - \frac{m}{y_n}) > 0$ and $V_j^0 = M - f(r_j) \geq 0$. Then by Lemma 6 we get

$$V_j^n \geq 0 , \quad \text{that is,} \quad u_j^n \leq M .$$

$u_j^n > 0$ follows directly from Lemma 6.

Next we shall prove that $\{- v_n\}$ is bounded above under the condition (2.2), that is,

$$0 \leq f(r) \leq N(\frac{1}{r} - \frac{1}{b}) \text{ (} N \text{ : positive constant) .}$$

Lemma 8 *Under the last condition we have*

$$(5.5) \qquad - \frac{2}{\alpha} \beta^{\frac{1}{2}} h^{\frac{1}{4}} \leq - v_n \leq \frac{N}{y_{n-1}y_n} < \frac{N}{b^2} \quad (n=1,2, \dots)$$

and

$$(5.6) \qquad 0 < y_n - y_{n-1} \leq (\frac{\alpha N}{b^2} + 1)k_n$$

and

$$(5.7) \qquad - \frac{2}{\alpha} \beta^{\frac{1}{2}} h^{\frac{1}{4}} (\frac{\alpha N}{b^2} + 1) \leq u_{\bar{t}}(r_{J+n-1}, t_n) < \frac{1}{b^2} (\frac{\alpha N}{b^2} + 1)(N + m), \quad (n=1,2, \dots) .$$

(Proof) We take a barrier function

$$\omega_{n_0}(r_j, t_n) = N[\frac{1}{r_j} - \frac{1}{y_{n_0}}] + \frac{m}{b} - \frac{m}{y_{n_0}}$$

Since, by Lemma 7

$$\omega_{n_0}(y_n, t_n) = N[\frac{1}{y_n} - \frac{1}{y_{n_0}}] + \frac{m}{b} - \frac{m}{y_{n_0}} > \frac{m}{b} - \frac{m}{y_n} \quad (0 \leq n < n_0)$$

$$\omega_{n_0}(r_j,0) = N[\frac{1}{r_j} - \frac{1}{y_{n_0}}] + \frac{m}{b} - \frac{m}{y_{n_0}} > N[\frac{1}{r_j} - \frac{1}{y_{n_0}}]$$

$$> N[\frac{1}{r} - \frac{1}{b}] \geq f ,$$

it follows from Lemma 7 that $\omega_{n_0} - u \geq 0$ $(0 \leq n \leq n_0)$ and hence

$$\frac{u(y_{n_0} - h,\ t_{n_0}) - (\frac{m}{b} - \frac{m}{y_{n_0}})}{h} \leq \frac{N}{(y_{n_0} - h)y_{n_0}}$$

which yields

$$- u_{\bar{r}}(y_{n_0},\ t_{n_0}) \leq \frac{N}{y_{n_0-1}\ y_{n_0}}$$

Since n_0 is arbitrary, we get the right parts of (5.5).

From the fact just above proved we get

$$\frac{y_n - y_{n-1}}{k_n} = -\frac{1}{2}(\alpha v_n - \sqrt{\alpha^2 v_n^{\ 2} + 4\beta\sqrt{h}}\ ) < \frac{\alpha N}{b^2} + 1$$

for small h and

$$- v_n = \frac{y_n - y_{n-1}}{\alpha\ k_n} - \frac{\beta\ k_n}{\alpha\sqrt{h}} > -\frac{2}{\alpha}\beta^{\frac{1}{2}} h^{\frac{1}{4}}$$

Noting that $u^{n-1}_{J+n-1} < u^{n}_{J+n}$ by Lemma 7 we get

$$u_{\bar{t}}^{\ n}(r_{J+n-1}, t_n) = \frac{u^{n}_{J+n-1} - u^{n-1}_{J+n-1}}{k_n} \geq \frac{u^{n}_{J+n-1} - u^{n}_{J+n}}{k_n}$$

$$= - \frac{h}{k_n} v_n \geq -\frac{2}{\alpha}\beta^{\frac{1}{2}} h^{\frac{1}{4}} (\frac{\alpha N}{b^2} + 1)$$

The function U_j^n defined by (5.4) satisfies

$$\frac{U^n_{J+n-1} - U^{n-1}_{J+n-1}}{k_n} = \frac{h}{k_n} \cdot \frac{U^n_{J+n-1} - U^n_{J+n}}{h} = \frac{h}{k_n}(-v_n) \le \frac{N}{b^2}\left(\frac{\alpha N}{b^2} + 1\right)$$

because of (4.5). Therefore

$$u_{\bar{t}}(r_{J+n-1}, t_n) = \frac{U^n_{J+n-1} - U^{n-1}_{J+n-1}}{k_n} + \frac{m}{k_n}\left(\frac{1}{y_{n-1}} - \frac{1}{y_n}\right)$$

$$< \frac{1}{b^2}\left(\frac{\alpha N}{b^2} + 1\right)(N + m) .$$

<u>Lemma 9</u> *Assume that in addition to (2.1), $f'(r)$ and $\Delta f(r)$ exist and $|f'(r)| \le M_1$, $|\Delta f(r)| \le M_2$ for $0 < r < b$ and $f(0) = 0$. Then it follows that*

$$|u_{\bar{r}}(r_j, t_n)| \le M_1$$

$$|u_{\bar{t}}(r_j, t_n)| \le M_2$$

for $0 < r_j < y_n$, $0 < t_n$.

(Proof) First note that $f(r)$ satisfies (2.2) with the appropriate constant $N = b_2 M_1$ (if it is necessary, we take M_1 sufficiently large). The function $W_j^n = M_1 \pm u_{\bar{r}}(r_j, t_n)$ satisfies the system of equations

$$\begin{cases} L_{0h} W_j^n = 0 , \\ W^n_{J+n} = M_1 \pm u_{\bar{r}}(y_n, t_n) \ge 0 , \\ W_1^n = 0 , \\ W_j^0 = M_1 \pm f_{\bar{r}}(r_j) \ge 0 \end{cases}$$

By the same argument in the proof of Lemma 6 we get $W_j^n \geq 0$, that is, $|u_{\bar{r}}(r_j,t_n)| < M_1$.

Next the function $Z_j^n = M_2 \pm u_{\bar{t}}(r_j,t_n)$ (if it is necessary, we take M_2 sufficiently large so that $M_2 > \frac{1}{b^2}(\frac{\alpha N}{b^2} + 1)(N + m))$ satisfies the system of equations

$$\begin{cases} L_{0h} Z_j^n = 0 \quad (n=2,3,4, \ldots), \\ Z_{J+n-1}^n = M_2 \pm u_{\bar{t}}(r_{J+n-1}, t_n) \geq 0 , \\ Z_1^n = Z_0^n , \\ Z_j^1 = M_2 \pm \Delta_h f(r_j) \geq 0 \end{cases}$$

Applying Lemma 6 we have $Z_j^n \geq 0$, that is

$$|u_{\bar{t}}(r_j,t_n)| \leq M_2 \quad \text{for} \quad 0 < r_j < y_n , \quad n=1,2, \ldots .$$

Thus we have proved Lemma 9.

Frow now we shall prove convergence of our difference scheme and as a byproduct existence of the solution of (P) again.

First of all we note that as $h \to 0$ k_n tends to zero uniformly by the fourth equation of (4.1). If we define the pieceweise linear boundary $y_h(t)$ by the formula

$$y_h(t) = \frac{t - t_n}{k_n} y_{n+1}(t) + \frac{t_{n+1} - t}{k_n} y_n(t) \quad \text{for} \quad t_n \leq t \leq t_{n+1},$$

$$(n=0,1,2, \ldots)$$

then by Lemma 4.3 $y_h(t)$ is Lipschitz continuous with a uniform Lipshitz constant $(\frac{\alpha N}{b^2} + 1)$ and uniformly bounded for $0 \leq t \leq T$. Therefore by Ascoli-Arzela's theorem there exists a subsequence of $\{y_h(t)\}$

(which we denote again by $\{y_h(t)\}$) such that as $h \to 0$ $y_h(t)$ uniformly converges to a Lipschitz continuous function $y(t)$.

Next it follows from Lemma 9 that for any positive a, $|u_{r\bar{r}}(r_j, t_n)|$ is also uniformly bounded for $a < r_j < y_n$ and $0 < t_n \leq T$. In fact it is clear by considering the first equation of (4.1). And further it can be seen that $\{u_{r\bar{r}r}\}$, $\{u_{r\bar{r}t}\}$ and $\{u_{\bar{t}r}\}$ are also uniformly bounded for $a < r_j < y(t) - \varepsilon$, $\varepsilon < t_n \leq T$ (ε, a are arbitrary small positive constant). Indeed if we put $(j - \frac{1}{2})h\, u_j^n = w_j^n$ we get the equation to be satisfied by w_j^n

$$(5.8) \qquad (w_j^n)_{r\bar{r}} - (w_j^n)_{\bar{t}} = 0$$

and we know that $\{w_j^n\}$ is uniformly bounded for $0 < r_j \leq y_n$, $0 \leq t_n \leq T$ since y_n is bounded for $0 \leq t_n \leq T$ by (5.6). Then it is well known that divided differens of high order of w_j^n with respect to t and r are all uniformly bounded for any closed region with finite distance from the boundary of the region $0 < r < y(t)$, $0 < t < T$ and for sufficiently small h (See Petrowsky's text book [10]), and hence that of u_j^n are so as. From these facts it follows that there exists a subsequence $\{u_j^n\}$ such that as $h \to 0$ u_j^n (in fact the extended ones over the whole region $0 < r < y(t)$, $t > 0$) converges to a infinitely differentiable function u uniformly for any closed region in $0 < r < y(t)$, $0 < t < T$ and $(u_j^n)_{r\bar{r}}$, $(u_j^n)_{\bar{t}}$ and $(u_j^n)_r$ converge uniformly to the corresponding derivatives $\frac{\partial^2 u}{\partial r^2}$, $\frac{\partial u}{\partial t}$ and $\frac{\partial u}{\partial r}$ respectively.

Let tend h to zero through the extracted subsequence in the first equation of (5.1), then the equation

$$\Delta u = \frac{\partial u}{\partial t}$$

holes in $0 < r < y(t)$, $0 < t \leq T$. Since $\{u_{\bar{t}}\}$ is uniformly bounded in the whole region by Lemma 9, the limit function $\frac{\partial u}{\partial t}$ and hence Δu are bounded in the whole region $0 < r < y(t)$, $0 < t \leq T$. Since $\frac{\partial u}{\partial r}$ is uniformly bounded in the whole region by the same reasoning from Lemma 9, there exist the limits $\lim_{r\to 0} u(r,t) = u(0,t)$ and $\lim_{r\to y(t)} u(r,t) = u(y(t),t)$ which is equal to $\frac{m}{b} - \frac{m}{y(t)}$ and hence $u(r,t)$ is continuous for $0 \leq r \leq y(t)$, $0 \leq t \leq T$. $\{u_{r\bar{r}}\}$ is also uniformly bounded for $b < r < y(t)$ and $0 < t < T$ from the first equation of (5.1) by Lemma 9. Therefore $\frac{\partial^2 u}{\partial r^2}$ is uniformly bounded for $b < r < y(t)$, $0 < t < T$ and hence there exists the limit $\lim_{r\to y(t)} \frac{\partial u}{\partial r} = v(t)$ (uniformly) and $v(t)$ is continuous for $0 < t \leq T$.

The fourth equation of (5.1) yields

$$y_n = b - \alpha \sum_{p=1}^{n} k_p v_p + \sum_{p=1}^{n} \beta \frac{k_p^2}{\sqrt{h}}$$

Let tend h to zero, then by noting that $\frac{k_p}{\sqrt{h}} < \frac{2h^{\frac{1}{4}}}{\sqrt{\beta}}$ the relation

$$y(t) = b - \alpha \int_0^t v(\tau)d\tau$$

holds. Hence $y(t)$ is differentiable and

$$\dot{y}(t) = -\alpha v(t)$$

Thus we have shown that the pair (y,u) of the limit function satisfies all the equations of (P) and therefore it is a solution of (P). It is easily seen from the uniqueness theorem (Theorem 1) that all the sequence $\{y_n, u_j^n\}_h$ converges to the solution (y,u).

Thus we have proved

Theorem 4 *Under the assumption of Lemma 9 there exist a solution* *(y,u) of (P), which is given as the limit of solution of our difference scheme (5.1) as $h \to 0$.*

6. Solving the difference scheme

In order to solve the nonlinear algebraic equation (5.1) at each time step $t = t_n$, we shall use the following iteration procedure

$$
(6.1)\quad \begin{cases}
k_n^{(0)} = k_{n-1} \quad (k_0 : \text{ arbitrary positive constant }), \\
u_{r\bar{r}}^{(s)}(r_j,t_n) + \dfrac{1}{r_j}\,(u_r^{(s)}(r_j,t_n) + u_{\bar{r}}^{(s)}(r_j,t_n)) \\
\qquad - \dfrac{u^{(s)}(r_j,t_n) - u(r_j,t_{n-1})}{k_n^{(s)}} = 0 , \\
u^{(s)}(r_1,t_n) = u^{(s)}(r_0,t_n), \\
u^{(s)}(r_{J+n},t_n) = \dfrac{m}{b} - \dfrac{m}{y_n} , \\
k_n^{(s+1)} = \dfrac{\sqrt{h}}{2\beta}\,(\alpha v_n^{(s)} + \sqrt{\alpha^2\{v_n^{(s)}\}^2 + 4\beta\sqrt{h}}), \\
\qquad (v_n^{(s)} = u_{\bar{r}}^{(s)}(y_n,t_n)) \quad (s=0,1,2, \dots) .
\end{cases}
$$

Convergence of this procedure will be proved for an appropriate constant β. For the purpose we put

$$
(6.2)\qquad r_j u^{(s)}(r_j,t_n) = w^{(s)}(r_j,t_n)
$$

and then we get the system to be satisfied by $w^{(s)}$:

$$
\begin{cases}
k_n^{(0)} = k_{n-1} , \\
w_{r\bar{r}}^{(s)}(r_j,t_n) - \dfrac{w^{(s)}(r_j,t_n) - w(r_j,t_n)}{k_n^{(s)}} = 0 ,
\end{cases}
$$

$$(6.3)\quad \begin{cases} w^{(s)}(r_1,t_n) + w^{(s)}(r_0,t_n) = 0 \\ w^{(s)}(r_{J+n},t_n) = y_n\left(\frac{m}{b} - \frac{m}{y_n} \right) \end{cases}$$

Now we introduce one step Green's function of the implicit difference analogue of heat equation :

$$g(r_j,\xi_\ell;\ t_n) = \frac{1}{2\pi h}\int_{-\pi}^{\pi} (1 + 4\lambda_n \sin^2\frac{\omega}{2})^{-1} \times$$

$$\times [e^{-i(r_j - \xi_\ell)\frac{\omega}{h}} - e^{-i(r_j + \xi_\ell)\frac{\omega}{h}}]d\omega\ ,$$

(6.4)

$$G(r_j,\xi_\ell;\ t_n) = \frac{1}{2\pi h}\int_{\pi}^{\pi} (1 + 4\lambda_n \sin^2\frac{\omega}{2})^{-1} \times$$

$$\times [e^{-i(r_j - \xi_\ell)\frac{\omega}{h}} + e^{-i(r_j + \xi_\ell - 1)\frac{\omega}{h}}]d\omega$$

where $\lambda_n = \frac{k_n}{h^2}$, $r_j = (j - \frac{1}{2})h$ and $\xi_\ell = (\ell - \frac{1}{2})h$. These functions satisfy the boundary conditions

$$(6.5)\quad \begin{aligned} &g(r_j,\xi_1;\ t_n) + g(r_j,\xi_0;\ t_n) = 0 \\ &g(r_1,\xi_\ell;\ t_n) + g(r_0,\xi_\ell;\ t_n) = 0 \end{aligned}$$

and conjugate relations

$$(6.6)\quad g_{\bar{r}} = - G_\xi \ , \quad G_r = - g_{\bar{\xi}}$$

By using the Green's function we can give a representation of $\{w^{(s)}(r_j,t_n)\}$: by using (6.5)

$$w^{(s)}(r_j,t_n^{(s)}) = \sum_{\ell=1}^{J+n-1} hg(r_j,\xi_\ell;\ t_n^{(s)})w(\xi_\ell,\ t_{n-1})$$

$$(6.7)\qquad + k_n^{(s)} g(r_j,y_n;\ t_n^{(s)})w_{\bar{r}}^{(s)}(y_n,t_n^{(s)}) -$$

$$- k_n^{(s)} g_{\bar{\xi}}(r_j, y_n;\ t_n^{(s)}) w^{(s)}(y_n, t_n^{(s)})$$

(Refer to [9]). Hence by (6.6)

$$w_{\bar{r}}^{(s)}(y_n, t_n) = - \sum_{\ell=1}^{J+n-1} hG_{\xi}(y_n, \xi_\ell;\ t_n^{(s)}) w(\xi_\ell,\ t_{n-1})$$
$$- k_n^{(s)}\ G_{\xi}(y_n, y_n;\ t_n^{(s)}) w_{\bar{r}}^{(s)}(y_n, t_n^{(s)})$$
$$+ k_n^{(s)}\ G_{\xi\bar{\xi}}(y_n, y_n;\ t_n^{(s)}) w^{(s)}(y_n, t_n^{(s)})$$

and

$$w_r^{(s)}(y_n, t_n) - w_r^{(s-1)}(y_n, t_n)$$
$$= [1 + k_n^{(s)} G_{\xi}(y_n, y_n;\ t_n^{(s)})]^{-1} [k_n^{(s)} G_{\xi\bar{\xi}}(y_n, y_n;\ t_n^{(s)}) w^{(s)}(y_n, t_n^{(s)})$$
$$- \sum_{\ell=1}^{J+n-1} hG_{\xi}(y_n, \xi_\ell;\ t_n^{(s)}) w(\xi_\ell, t_{n-1})]$$
$$- [1 + k_n^{(s-1)} G_{\xi}(y_n, y_n;\ t_n^{(s-1)})]^{-1} [k_n^{(s-1)} G_{\xi\bar{\xi}}(y_n, y_n;\ t_n^{(s-1)}) \times$$
$$\times\ w^{(s-1)}(y_n, t_n^{(s-1)}) - \sum_{\ell=1}^{J+n-1} hG_{\xi}(y_n, \xi_\ell;\ t_n^{(s-1)}) w(\xi_\ell, t_{n-1})]$$

It can be proved in the same way as in § 3 of [9] that

$$|w_{\bar{r}}^{(s)}(y_n, t_n) - w_{\bar{r}}^{(s-1)}(y_n, t_n)| < \frac{\text{const.}}{\beta} |\ w_{\bar{r}}^{(s-1)}(y_n, t_n) - w_{\bar{r}}^{(s-2)}(y_n, t_n)|$$

and hence for a large β $\{w_r^{(s)}(y_n, t_n)\}$ converges to the limit $w_{\bar{r}}(y_n, t_n)$ as $s \to \infty$ and therefore $\{v_n^{(s)}\}$, $\{k_n^{(s)}\}$ converge to the limits v_n and k_n respectively. And further by (6.7) $\{w^{(s)}(r_j, t_n^{(s)})\}$ converges to the limit $w(r_j, t_n)$. Then it is clear that $\{u(r_j, t_n) \equiv \frac{1}{r_j}\ w(r_j, t_n)\}$ $(j=0,1,2, \ldots J+n)$ and k_n satisfy (4.1) at t_n. Thus we have proved that we can take constant β so large that our iteration procedure converges.

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