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Caratheodory-type equations in a Banach space

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We consider in a real Banach space E the initial value problem of an ordinary differential equation of Carathéodory type:

(1) $du/dt = f(t,u), \quad u(0) = a.$

We assume the following conditions are satisfied.

(I) For each $x \in E$, the function $f(.,x)$: $I \to E$ is strongly measurable, where $I = [0,T]$ for a constant $T > 0$. For each $t \in I$, the function $f(t,.)$ on E_s with the strong topology into E_w with the weak topology is continuous.

(II) For each $\rho > 0$, there exists $\gamma_\rho \in L^1(I;\mathbb{R})$ such that

$$|f(t,x)| \leqslant \gamma_\rho(t)$$

whenever $t \in I$ and $|x| \leqslant \rho$.

In case E is finite-dimesional and f is continuous with respect to (t,x), Okamura [3] showed the following necessary and sufficient condition for the uniqueness of solutions of the initial value problem (1) : If f is a contionuous function on a domain $D \subset \mathbb{R}^{N+1}$ into $\mathbb{R}^N$, a necessary and sufficient condition for the uniqueness of solutions of $du/dt = f(t,u)$, existing to the right of the initial value (t_0,u_0) for each $(t_0,u_0) \in D$, is that there exists a $\Phi \in C^1(\hat{D};\mathbb{R})$, $\hat{D} = \{(t,x_1,x_2);\ (t,x_i) \in D, i=1,2\}$, such that

(2) $\Phi(t;x_1,x_2) \geqslant 0, \quad \Phi(t;x_1,x_2) = 0 \Leftrightarrow x_1 = x_2$

and

$$(3) \qquad \frac{\partial \Phi}{\partial t}(t;x_1,x_2) + \sum_{j=1}^{2} \langle f(t,x_j), \mathrm{grad}_{x_j}\Phi(t;x_1,x_2)\rangle \leqslant 0$$

hold.

It may be natural that we introduce the Okamura's function Φ in our case. Acturally Murakami [2] made use of the Okamura's function Φ in the case of Banach spaces to give a sufficient condition for the existence and uniqueness of solutions of the initial value problem (1) when f is continuous with respect to (t,x). Also, Murakami [2] mentioned Caretheódory-type equations, but he did not enter into details. So we shall treat Carathéodory-type equations in detail to some extent.

Let E be a real Banach space. Let $\Phi(t;x,y)$ be a real-valued function defined on $I \times E \times E$. Suppose Φ satisfies the following conditions (i)-(iii):

(i) $\quad \Phi(t;x,y) \geqslant 0$ and $\Phi(t;x,x) = 0$

for all $t \in I$ and all $x,y \in E$.

(ii) For any $t \in I$ and any $\rho > 0$, if two sequences $\{x_n\}$ and $\{y_n\}$ in $B_\rho \equiv \{z \in E; |z| \leqslant \rho\}$ satisfies $\Phi(t;x_n,y_n) \to 0$ as $n \to \infty$, then $|x_n - y_n| \to 0$ as $n \to \infty$.

(iii) For any $\rho > 0$, there exist a $\beta_\rho \in L^1(I;\mathbb{R})$ and a positive constant C_ρ such that

$$|\Phi(t_1;x_1,y_1) - \Phi(t_2;x_2,y_2)| \leqslant \int_{t_1}^{t_2} \beta_\rho(s)\,ds + C_\rho(|x_1-x_2|+|y_1-y_2|)$$

holds whenever $t_1,t_2 \in I$, $t_1 \leqslant t_2$, and x_j, $y_j \in B_\rho$, $j=1,2$.

We define

$$\Phi'_{\pm}(t;x,y;a,b)$$

$$= \liminf_{\xi \to \pm 0} \{\Phi(t+\xi;x+\xi a,y+\xi b) - \Phi(t;a,b)\}/\xi$$

for all $t \in I$ and all $x,y,a,b \in E$.

<u>Example.</u> Let $\beta \in L^1(I;\mathbb{R})$. If we set

$$\Phi(t;x,y) = |x - y|^2 \exp(-2\int_0^t \beta(s)\,ds) \tag{4}$$

for all $x,y \in E$ and a.a. $t \in I$, then Φ satisfies (i)-(iii) and

$$\Phi'_{\pm}(t;x,y;a,b) = 2\{\langle a-b,x-y\rangle_{\pm} - \beta(t)|x-y|^2\} \exp(-2\int_0^t \beta(s)\,ds) \tag{5}$$

holds for all x,y,a and $b \in E$ and a.a. $t \in I$. Here $\langle .,. \rangle_{\pm}$ is defined by

$$\langle u,v\rangle_{\pm} = \pm\{\sup \pm\langle u,v^*\rangle;\ v^* \in F(v)\}$$

for all $u,v \in E$, where F is the duality map of E into its dual E'.

We are concerned with the following condition imposed on Φ and f:

(III) $\Phi'_{-}(t;x,y;f(t,x),f(t,y)) \leq 0$

holds for a.a. $t \in I$ and all $x,y \in E$.

<u>Example.</u> If Φ is given by (4), then (III) is reduced to the following condition:

$$\langle f(t,x) - f(t,y),x - y\rangle_{-} \leq \beta(t)|x - y|^2$$

holds for a.a. $t \in I$ and all $x,y \in E$. See (5). Hence, if $\beta = 0$, (III) is equivalent to the condition:

(III') $-f(t,.)$ is accretive for a.a. $t \in I$.

Now we shall prove a main result.

Theorem 1. Let E be an arbitrary real Banach space. Suppose f satisfies (I)-(III) with a Φ having properties (i)-(iii) and, moreover, for each $t \in I$, the function $f(t,.) : E_s \to E_s$ is locally uniformly continuous. Then, for any $a \in E$, there exist an $r \in (0,T]$ and a unique $u \in C([0,r];E)$ such that

$$(6) \qquad u(t) = a + \int_0^t f(s,u(s))ds$$

holds for all $t \in [0,r]$.

Proof. Let $a \in E$ and $\rho > 0$. By (II) there exists a function $\gamma \in L^1(I;\mathbf{R})$ such that

$$|f(t,x)| \leqslant \gamma(t)$$

whenever $t \in I$ and $|x-a| \leqslant \rho$. We choose an $r \in (0,T]$ satisfying $\int_0^r \gamma(t)dt \leqslant \rho$. For each integer $n > 1/r$, we define

$$(7) \qquad u_n(t) = \begin{cases} a & (t \leqslant 1/n) \\ a + \int_{\frac{1}{n}}^t f(s,u_n(s-n^{-1}))ds & (1/n \leqslant t \leqslant r). \end{cases}$$

Clearly we have $|u_n(t) - a| \leqslant \rho$ whenever $n > 1/r$ and $t \leqslant r$. Let $m,n > 1/r$. By using (iii), we can verify easily that $\Phi(t;u_m(t),u_n(t))$ is absolutely continuous on $0 \leqslant t \leqslant r$. Hence we have

$$(8) \qquad \Phi(t;u_m(t),u_n(t)) = \int_0^t \frac{d}{ds}\Phi(s;u_m(s),u_n(s))ds$$

for $t \in [0,r]$. Using (iii) again, we obtain

$$\frac{d}{ds}\Phi(s;u_m(s),u_n(s))$$

$$= \frac{d}{d\xi}\Phi(s+\xi; u_m(s)+\xi\cdot f(s,u_m(s-m^{-1})), u_n(s)+\xi\cdot f(s,u_n(s-n^{-1})))\Big|_{\xi=0}$$

for almost all s if $du_k(s)/ds = f(s,u_k(s-k^{-1}))$ exists for k=m,n. Hence, by (III), we have

$$\frac{d}{ds}\Phi(s;u_m(s),u_n(s))$$

$$\leq \Phi'_-(s;u_m(s),u_n(s);f(s,u_m(s-m^{-1})),f(s,u_n(s-n^{-1})))$$

$$- \Phi'_-(s;u_m(s),u_n(s);f(s,u_m(s)),f(s,u_n(s)))$$

$$\equiv A(s;m,n)$$

for a.a. $s \in (\max(m^{-1},n^{-1}),r)$. Then, by the assumption of the locally uniform continuity of $f(s,.)$,

$$\lim_{m,n\to\infty} A(s;m,n) = 0$$

holds for almost all s. Then, by using (ii) and applying the Lebesgue-Fatou theorem to (8) and (7), it is verified that $u(t) = \lim_{n\to\infty} u_n(t)$ exists for all $t \in [0,r]$ and u satisfies (6).

The uniqueness of solutions is shown by a standard argument.

We can prove the following result in the same manner as we proved the preceding theorem, noticing that the duality map F is single-valued and locally uniformly continuous if the dual space E' of E is uniformly convex. See Kato[1], Lemma 1.2.

Theorem 2. Suppose E' is uniformly convex and f satisfies (I)-(III). Then the conclusion of Theorem 1 holds, if E is separable.

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References

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