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TAIL PROBABILITIES OF SOME CONTINUOUS FUNCTIONALS OF GAUSSIAN PROCESSES

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1. Let $X = \{X(t),\ 0 \le t \le 1\}$ be a path continuous Gaussian process with mean zero, and let T be a real continuous functional on $C[0,1]$ such that $T(cx) = c^p T(x)$ with $p > 0$ for any positive constant c. In this note the following asymptotic estimate for the tail probabilities of $T(X)$ is obtained:

$$\lim_{\alpha\to\infty} (1/\alpha^{2/p})\cdot\log P\{\ T(X) > \alpha\ \} = -(1/2)b^2,$$

where b^2 is a constant determined as the solution of certain extremal problem. For example, it is shown that if X is Brownian motion, then

$$\lim_{\alpha\to\infty} (1/\alpha^{2/p})\cdot\log P\{\ \int_0^1 |X(t)|^p dt > \alpha\ \} = -(1/2)(c(p))^{-2/p},$$

where $p \ge 1$ and

$$c(p) = 2(p+2)^{(p/2)-1}/(\int_0^1 (1-t^p)^{-1/2} dt)^p p^{p/2},$$

and also, if X is Brownian bridge, then the same formula holds with $c(p)$ replaced by $2^{-p}c(p)$.

In his thesis [3] and also in [4], N. A. Marlow obtained a similar asymptotic formula for tail probabilities of uniformly Hölder continuous, asymptotically homogeneous functionals F of path continuous Gaussian processes. His method of proof is to first estimate $\log P\{ F(X) > \alpha \}$ in the finite-dimensional case by a Laplace asymptotic formula, and then to pass to the limit to obtain the function space version. Note also that H. P. McKean [5] obtained a similar asymptotic estimate for tail probabilities of multiple Wiener integrals.

Our method is different from Marlow's and is based on the following Fredlin-Wentzell type estimates for Gaussian measures given in [7] and [2]. Let $C = C[0,1]$ be the space of all continuous functions on $[0,1]$ with the supremum norm $||\cdot||_\infty$, and let $\mathcal{A}$ be the σ-field of Borel subsets of C. Let μ be a Gaussian measure on $(C, \mathcal{A})$ with mean zero and covariance function $R(s,t)$, i.e., $\int_C x(t)\mu(dx) = 0$, for $0 \le t \le 1$, and $R(s,t) = \int_C x(s)x(t)\mu(dx)$, for $0 \le s, t \le 1$, where $x \in C$. Let $H = H(R)$ be the reproducing kernel Hilbert space (RKHS) with reproducing kernel (r.k.) R, whose norm is denoted by $||\cdot||_H$. Note that $H \subset C$, since R is continuous.

<u>Theorem 1</u>. Let $\phi \in H$. Then, for any $\delta, h > 0$, there is a number $\alpha_0 = \alpha_0(\delta, h, ||\phi||_H)$ such that

$$\mu\{ x \mid ||(x/\alpha) - \phi||_\infty < \delta \} \ge \mu\{ x \mid ||x - \alpha\phi||_\infty < \delta \}$$
$$\ge \exp[-(\alpha^2/2)(||\phi||_H^2 + h)]$$

for all $\alpha \ge \alpha_0$.

<u>Theorem 2</u>. Let $K_r = \{ \phi \in H \mid ||\phi||_H \le r \}$ and let $d(x, K_r)$ be the distance from $x \in C$ to K_r in the sup norm $||\cdot||_\infty$. Then, for any $\delta, h > 0$,

there is a number $\alpha_0 = \alpha_0(\delta, h, r)$ such that

$$\mu\{\, x \mid d(x/\alpha, K_r) > \delta \,\} \le \exp[-(\alpha^2/2)(r^2 - h)]$$

for all $\alpha \ge \alpha_0$.

For the proofs see [7] or [2]. From Theorems 1 and 2 we obtain the following

Theorem 3. Let T be a real continuous functional on C such that $T(cx) = c^p T(x)$ with $p > 0$ for any positive constant c and $T(\phi) > 0$ for some $\phi \in H$. Then

$$\lim_{\alpha\to\infty} (1/\alpha^{2/p})\cdot\log \mu\{\, x \mid T(x) > \alpha \,\} = -(1/2)b^2,$$

where $b^2 = \inf\{\, ||\phi||_H^2 \mid T(\phi) > 1 \,\} = \sup\{\, r^2 \mid \sup\{T(\phi) \mid \phi \in K_r\} < 1 \,\}$.

Proof. Let $D = \{\, x \mid T(x) > 1 \,\}$. D is open and its closure $\overline{D} = \{\, x \mid T(x) \ge 1 \,\}$. For any $\phi \in H \cap D$, there is a $\delta > 0$ such that $||x - \phi||_\infty < \delta$ implies $x \in D$. Hence, using Theorem 1, we obtain

$$\begin{aligned}\mu\{\, x \mid T(x) > \alpha \,\} &= \mu\{\, x \mid T(x/\alpha^{1/p}) > 1 \,\}\\ &\ge \mu\{\, x \mid ||(x/\alpha^{1/p}) - \phi||_\infty < \delta \,\}\\ &\ge \exp[-(\alpha^{2/p}/2)(||\phi||_H^2 + h)]\end{aligned}$$

for any $h > 0$, if α is sufficiently large. Thus, for any $\phi \in H \cap D$,

$$\liminf_{\alpha\to\infty} (1/\alpha^{2/p})\cdot\log \mu\{\, x \mid T(x) > \alpha \,\} \ge -(1/2)||\phi||_H^2,$$

and hence,

$$\liminf_{\alpha\to\infty} (1/\alpha^{2/p})\cdot\log \mu\{ x \mid T(x) > \alpha \}$$

$$\geq -(1/2)\cdot\inf \{ ||\phi||_H^2 \mid T(\phi) > 1 \}.$$

Since K_r is compact in C (see, e.g. [6]) and T is continuous, there is a number $r > 0$ such that $\sup\{T(\phi) \mid \phi \in K_r\} < 1$, and for any such a number r, there is a $\delta > 0$ such that $d(K_r, \overline{D}) > \delta$, where $d(K_r, \overline{D})$ is the distance between K_r and $\overline{D}$. If $T(x) > \alpha$, then $x/\alpha^{1/p} \in D$, and by Theorem 2,

$$\mu\{ x \mid T(x) > \alpha \} \leq \mu\{ x \mid d(x/\alpha^{1/p}, K_r) > \delta \}$$

$$\leq \exp[-(\alpha^{2/p}/2)(r^2 - h)]$$

for any $h > 0$, if α is sufficiently large. Therefore,

$$\limsup_{\alpha\to\infty} (1/\alpha^{2/p})\cdot\log \mu\{ x \mid T(x) > \alpha \} \leq -(1/2)r^2,$$

and hence

$$\limsup_{\alpha\to\infty} (1/\alpha^{2/p})\cdot\log \mu\{ x \mid T(x) > \alpha \}$$

$$\leq -(1/2)\cdot\sup\{ r^2 \mid \sup\{T(\phi) \mid \phi \in K_r\} > 1 \}.$$

It is easy to see that $\inf\{ ||\phi||_H^2 \mid T(\phi) > 1 \} = \inf\{ ||\phi||_H^2 \mid T(\phi) \geq 1 \}$ (in fact, $= \inf\{ ||\phi||_H^2 \mid T(\phi) = 1 \}$), and if $r^2 < \inf\{ ||\phi||_H^2 \mid T(\phi) \geq 1 \}$, then $\sup\{ T(\phi) \mid \phi \in K_r \} < 1$, and so $\sup\{ r^2 \mid \sup\{T(\phi) \mid \phi \in K_r\} < 1 \} = \inf\{ ||\phi||_H^2 \mid T(\phi) > 1 \}$. This completes the proof.

Remark. Note that $\sup\{T(\phi) \mid \phi \in K_b\} = 1$. Since $\sup\{T(\phi) \mid \phi \in K_b\} = b^p\cdot\sup\{T(\phi) \mid \phi \in K_1\}$, we have $b^2 = (\sup\{T(\phi) \mid \phi \in K_1\})^{-2/p}$.

2. In what follows we consider several examples for which the values of b^2 can be explicitely given by evaluating $\sup\{T(\phi) \mid \phi \in K_1\}$.

(i) Let X be a path continuous Gaussian process with mean zero and covariance function R(s,t). Then

$$\lim_{\alpha\to\infty} (1/\alpha)\cdot\log P\{ \int_0^1 X^2(t)\,dt > \alpha \} = -1/(2\lambda_1),$$

where λ_1 is the largest eigenvalue of the covariance operator R with kernel R(s,t) on $L^2[0,1]$.

This is a known result, and so we just indicate briefly how it can be derived from Theorem 3. In this case $T(x) = \int_0^1 x^2(t)\,dt = ||x||_2^2$ and $p = 2$. Let $\{\lambda_i\}$ and $\{\psi_i\}$ be the eigenvalues and the corresponding normalized eigenfunctions of R. Then $\{\phi_i = \lambda_i^{1/2}\psi_i\}$ is a complete orthonormal system in H(R). It can be shown that $||\phi||_2^2 \le \lambda_1 ||\phi||_H^2$ for any $\phi \in H(R)$. Hence $\sup\{T(\phi) \mid \phi \in K_1\} \le \lambda_1$. Since $||\phi_1||_2^2 = \lambda_1$, we have $\sup\{T(\phi) \mid \phi \in K_1\} = \lambda_1$, and hence the result.

(ii) Let μ be the Wiener measure and let $T(x) = \int_0^1 |x(t)|^p dt$, $p \ge 1$. The RKHS H(R) associated with the Wiener measure is the space of all absolutely continuous functions ϕ on [0,1] such that $\phi(0) = 0$ and $d\phi/dt \in L^2[0,1]$, and $(\phi, \psi)_H = \int_0^1 (d\phi/dt)(d\psi/dt)\,dt$, where $(\cdot,\cdot)_H$ denotes the inner product of H(R). V. Strassen ([8], p.220) proved that $\sup\{T(\phi) \mid \phi \in K_1\} = c(p)$, where

$$c(p) = 2(p+2)^{(p/2)-1}/(\int_0^1 (1-t^p)^{-1/2} dt)^p p^{p/2}.$$

We thus obtain the result for Brownian motion stated at the beginning of this note. In particular, $C(1) = 3^{-1/2}$ and $c(2) = 4/\pi^2$. The case $p = 1$ has been previously obtained by Marlow [3] by a different method, and the case p = 2 is of course a particular case of (i). If p in an integer, then the same formula holds for $T(x) = \int_0^1 (x(t))^p dt$.

(iii) Let μ be the Wiener measure and let

$$T(x) = \int_0^1 |x(t)|^2 dt / \int_0^1 |x(t)| dt.$$

Then $\sup\{T(\phi) \mid \phi \in K_1\} = 2q$, where $0 < q < 1$ is the largest solution of

$$(1-q)^{1/2} \sin((1-q)^{1/2}/q) + \cos((1-q)^{1/2}/q) = 0$$

(see [8], p.222). Hence, if X is Brownian motion, then

$$\lim_{\alpha\to\infty} (1/\alpha^2)\cdot\log P\{ \int_0^1 |X(t)|^2 dt / \int_0^1 |X(t)| dt > \alpha \} = -1/(8q)^2.$$

(iv) Let X be Brownian bridge. We shall show that

$$\lim_{\alpha\to\infty} (1/\alpha^{2/p})\cdot\log P\{ \int_0^1 |X(t)|^p dt > \alpha \} = -2(c(p))^{-2/p}, \quad p \geq 1,$$

where c(p) is the same as in (ii).

The covariance function of Brownian bridge is

$$R(s,t) = \begin{cases} s(1-t), & \text{for } s \leq t, \\ t(1-s), & \text{for } s \geq t, \end{cases}$$

$$= \int_0^1 Q(u,s)Q(u,t)du,$$

where

$$Q(u,t) = \begin{cases} 1-t, & \text{for } u \leq t, \\ -t\ , & \text{for } u > t. \end{cases}$$

Hence the RKHS H(R) with r.k. R is isometrically isomorphic to the closed subspace M of $L^2[0,1]$, spanned by $\{Q(u,t),\ 0 \leq t \leq 1\}$, and any function ϕ in H(R) has a representation $\phi(t) = \int_0^1 m(u)Q(u,t)du$ with $m \in M$. Note that

$M \perp 1$, i.e., $\int_0^1 m(u)du = 0$ for all $m \in M$, since $\int_0^1 Q(u,t)du = 0$ for all $t \in$ $[0,1]$ and if $\int_0^1 n(u)du = 0$ and $\int_0^1 n(u)Q(u,t)du = 0$ for all $t \in [0,1]$, then $n = 0$. Hence $\phi(t) = \int_0^1 m(u)Q(u,t)du = \int_0^t m(u)du$, which shows that ϕ is absolutely continuous. Therefore, $H(R)$ is the space of all absolutely continuous functions ϕ on $[0,1]$ such that $\phi(0) = \phi(1) = 0$ and $\phi' = d\phi/dt \in L^2[0,1]$, and $(\phi,\psi)_H = \int_0^1 \phi'\psi' dt$.

As in Strassen's proof [8] for Brownian motion case, we shall evaluate $\sup\{T(\phi) \mid \phi \in K_1\} = \sup\{ \int_0^1 |\phi(t)|^p dt \mid \phi(0) = \phi(1) = 0 \text{ and } \int_0^1 \phi'^2 dt \le 1 \}$ by classical methods of the calculus of variations. Since K_1 is compact and T is continuous, there is a maximizing point ϕ with $||\phi||_H^2 = \int_0^1 \phi'^2 dt = 1$. We may assume $\phi \ge 0$, and ϕ satisfies the equation

$$\int_0^1 p\phi^{p-1}\psi dt = 2\lambda \cdot \int_0^1 \phi'\psi' dt, \quad \text{for any } \psi \in H(R),$$

where $\lambda > 0$ is a Lagrange multiplier. Integrating by parts the left-hand side and noting that $\psi' \perp 1$, we obtain

$$\int_0^1 \{\int_t^1 p\phi^{p-1}(s)ds - \int_0^1 [\int_s^1 p\phi^{p-1}(u)du]ds\}\psi'(t)dt = 2\lambda \cdot \int_0^1 \phi'\psi' dt$$

for all $\psi' \in M$. Therefore,

$$(1) \qquad \int_t^1 p\phi^{p-1}(s)ds - \int_0^1 [\int_s^1 p\phi^{p-1}(u)du]ds = 2\lambda\phi'(t), \quad \text{for } 0 \le t \le 1.$$

Since $\phi \ge 0$ and $\lambda > 0$, (1) shows that $\phi'(0) \ge 0$, $\phi'(1) \le 0$ and ϕ' is differentiable and monotone decreasing. Hence there is a point t_0 such that $\phi'(t_0) = 0$ and $\phi'(t) \ge 0$ or ≤ 0 according as $0 \le t \le t_0$ or $t_0 \le t \le 1$. Differentiating (1), multiplying with ϕ' and integrating again, we have

$$(2) \qquad \phi^p(t) + \lambda\phi'^2(t) = \phi^p(1) + \lambda\phi'^2(1) = \lambda\phi'^2(1).$$

Hence $|\phi'(1)| > 0$ and

$$\phi'(t) = \begin{cases} (\phi'^2(1) - (1/\lambda)\phi^p(t))^{1/2} & \text{for } 0 \le t \le t_0, \\ -(\phi'^2(1) - (1/\lambda)\phi^p(t))^{1/2} & \text{for } t_0 \le t \le 1. \end{cases}$$

Therefore, noting that $\phi(o) = \phi(1) = 0$, we get, for $0 \le t \le t_0$,

$$(3) \qquad t = \int_0^{\phi(t)} |\phi'(1)|^{-1}(1 - u^p/(\lambda\phi'^2(1)))^{-1/2}du$$

$$= \lambda^{1/p}|\phi'(1)|^{(2/p)-1}\int_0^{\phi(t)/(\lambda\phi'^2(1))^{1/p}}(1 - v^p)^{-1/2}dv,$$

and, for $t_0 \le t \le 1$,

$$(4) \qquad t - 1 = -\lambda^{1/p}|\phi'(1)|^{(2/p)-1}\cdot\int_0^{\phi(t)/(\lambda\phi'^2(1))^{1/p}}(1 - v^p)^{-1/2}dv.$$

Put $t = t_0$ in (3) and (4). Then $t_0 = 1 - t_0$, and so $t_0 = 1/2$. Put $t = 1/2$ in (2) and (3). Then $\phi^p(1/2) = \lambda\phi'^2(1)$ and

$$(5) \qquad 1/2 = \lambda^{1/p}|\phi'(1)|^{(2/p)-1}\cdot\int_0^1(1 - v^p)^{-1/2}dv.$$

Integrating (2) and noting $\int_0^1\phi'^2dt = 1$, we have

$$(6) \qquad \int_0^1\phi^p(t)dt = \lambda(\phi'^2(1) - 1)$$

Using (3) and (4), we obtain

$$\int_0^{1/2}\phi^p(t)dt = \lambda^{1+)1/p)}|\phi'(1)|^{1+(2/p)}\cdot\int_0^1 v^p(1 - v^p)^{-1/2}dv$$

$$= \int_{1/2}^1\phi^p(t)dt.$$

Hence

$$(7) \qquad \int_0^1\phi^p(t)dt = 2\lambda^{1+(1/p)}|\phi'(1)|^{1+(2/p)}\cdot\int_0^1 v^p(1 - v^p)^{-1/2}dv$$

$$= 4(p+2)^{-1}\lambda^{1+(1/p)}|\phi'(1)|^{1+(2/p)}\cdot\int_0^1(1 - v^p)^{-1/2}dv.$$

Eliminating λ and $|\phi'(1)|$ from (5), (6) and (7), we obtain

$$\int_0^1 \phi^p(t)\,dt = 2^{1-p}(p+2)^{(p/2)-1}/(\int_0^1 (1 - t^p)^{-1/2}dt)^p p^{p/2} = 2^{-p}c(p).$$

Thus $b^2 = 4(c(p))^{-2/p}$.

Remark. If p is an integer, $T(x) = \int_0^1 |x(t)|^p dt$ can be replaced by $T(x) = \int_0^1 x^p(t)\,dt$. The above result $\sup\{T(\phi) \mid \phi \in K_1\} = 2^{-p}c(p)$ can be used to obtain an iterated logarithm result for the functional T of empirical distributions (cf. H. Finkelstein [1]). Finkelstein discusses only the case p = 2, which can be obtained as a particular case of (i).

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