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ON THE AVERAGE ORDER OF THE SUM $\sum_{p \leqq x} \left(\frac{p}{q}\right)$

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Throughout in this note p, q will designate odd prime numbers; (m/n) is the Legendre-Jacobi symbol (with n odd).

It is proved by H. Heilbronn (On the averages of some arithmetical functions of two variables, Mathematika 5 (1958), 1 - 7) that

$$\sum_{p \leqq x} \sum_{q \leqq x} \left(\frac{p}{q}\right) = O\left(x^{\frac{7}{4}} (\log x)^{-\frac{5}{4}}\right) \qquad (x > 2),$$

so that we have in a sense

$$(*) \qquad \text{average order} \sum_{p \leqq x} \left(\frac{p}{q}\right) = O\left(x^{\frac{3}{4}} (\log x)^{-\frac{1}{4}}\right)$$

(and similarly for 'average order $\sum_{q \leqq x} \left(\frac{p}{q}\right)$ ').

The main purpose of this note is to improve (*) to a certain extent, within the logarithmic factor.

We shall first examine the sum

$$\sum_{p \leqq x} \sum_{q \leqq y} \left(\frac{p}{q}\right) \qquad (x,\ y > 2).$$

Following the argument of Heilbronn, we have

$$\left(\sum_{p\leq x}\sum_{q\leq y}\left(\frac{p}{q}\right)\right)^2 \leq \sum_{q\leq y} 1\cdot \sum_{q\leq y}\left(\sum_{p\leq x}\left(\frac{p}{q}\right)\right)$$

$$\leq \pi(y)\sum_{p_1}\sum_{p_2}\sum_{q}\left(\frac{p_1p_2}{q}\right) ,$$

where $\pi(t)$ denotes the number of primes $p \leq t$, and

$$\left(\sum_{p_1}\sum_{p_2}\sum_{q}\left(\frac{p_1p_2}{q}\right)\right)^2 \leq \sum_{p_1}\sum_{p_2} 1\cdot 2 \sum_{1\leq n\leq x^2}\left(\sum_{q}\left(\frac{n}{q}\right)\right)^2$$

$$\leq 2\pi^2(x)\sum_{q_1}\sum_{q_2}\sum_{n}\left(\frac{n}{q_1q_2}\right) .$$

Now

$$\sum_{1\leq n\leq x^2}\left(\frac{n}{q_1q_2}\right) \leq x^2 \qquad \text{if} \quad q_1 = q_2 ;$$

$$\sum_{1\leq n\leq x^2}\left(\frac{n}{q_1q_2}\right) \leq (q_1q_2)^{\frac{1}{2}} \log q_1q_2 \ll y \log y \quad \text{if} \quad q_1 \neq q_2,$$

so that

$$\sum_{q_1,\ q_2}\sum_{n}\left(\frac{n}{q_1q_2}\right) \ll \pi(y)x^2 + \pi^2(y)y \log y .$$

Therefore

$$\left(\sum_{p\leq x}\sum_{q\leq y}\left(\frac{p}{q}\right)\right)^4 \ll \pi^2(x)\pi^2(y)(x^2\pi(y) + \pi^2(y)y \log y)$$

$$\ll \frac{x^4}{\log^2 x}\,\frac{y^3}{\log^3 y} + \frac{x^2}{\log^2 x}\,\frac{y^5}{\log^3 y} .$$

By virtue of the quadratic reciprocity law the above argument can be repeated with p, q interchanged by each other, thus giving

$$\left(\sum_{p\leqq x}\ \sum_{q\leqq y}\left(\frac{p}{q}\right)\right)^4 \ll \frac{x^3}{\log^3 x}\,\frac{y^4}{\log^2 y} + \frac{x^5}{\log^3 x}\,\frac{y^2}{\log^2 y}\ .$$

Hence:

$$\sum_{p\leqq x}\ \sum_{q\leqq y}\left(\frac{p}{q}\right) = \begin{cases} O\left(\dfrac{x}{\log^{\frac{1}{2}} x}\,\dfrac{y^{\frac{3}{4}}}{\log^{\frac{3}{4}} y}\right) & \text{if} \quad x \geqq y\ , \\[2ex] O\left(\dfrac{x^{\frac{3}{4}}}{\log^{\frac{3}{4}} x}\,\dfrac{y}{\log^{\frac{1}{2}} y}\right) & \text{if} \quad x \leqq y\ . \end{cases}$$

Next, let r be a fixed positive integer and consider the sum

$$S_r = S_r(x,\ y) = \sum_{q\leqq y}\left(\sum_{p\leqq x}\left(\frac{p}{q}\right)\right)^r \qquad (x,\ y > 2).$$

We have

$$S_r = \sum_{q\leqq y}\ \sum_{p_1,\ldots,p_r\leqq x}\left(\frac{p_1\cdots p_r}{q}\right)\ ;$$

$$S_r^2 \leqq \sum_{p_1,\ldots,p_r} 1\cdot \sum_{p_1,\ldots,p_r}\left(\sum_{q}\left(\frac{p_1\cdots p_r}{q}\right)\right)^2$$

$$\leq r!\ \pi^r(x)\cdot \sum_{q,q'} \sum_{1\leq n\leq x^r} \left(\frac{n}{qq'}\right)$$

$$\ll \pi^r(x)(\pi(y)x^r + \pi^2(y)y \log y)$$

$$\ll \frac{x^{2r}}{\log^r x}\,\frac{y}{\log y} \qquad \text{if} \qquad x^r \geq y^2.$$

Putting $y = x^{r/2}$ (> 2), we find

$$\text{average order} \sum_{p\leq x}\left(\frac{p}{q}\right) \ll \frac{x}{\log^{\frac{1}{2}} x}\,\frac{\log^{\frac{1}{2r}} y}{y^{\frac{1}{2r}}} = \frac{x^{\frac{3}{4}}}{(\log x)^{\frac{1}{2}-\frac{1}{2r}}}.$$

Since r may be chosen arbitrarily large, this proves that

$$\text{average order} \sum_{p\leq x}\left(\frac{p}{q}\right) = O_\varepsilon(x^{\frac{3}{4}}(\log x)^{-\frac{1}{2}+\varepsilon})$$

for any fixed $\varepsilon > 0$.

NOTE. As has been suggested by Mr. T. Hirano, our argument applies of course to the sums of the form

$$T_r = \sum_{y_1<q\leq y}\left(\sum_{x_1<p\leq x}\left(\frac{p}{q}\right)\right)^r \qquad (r \geq 1),$$

yielding e.g. an estimate

$$T_r^2 \ll (\pi(x) - \pi(x_1))^r \cdot$$

$$((\pi(y) - \pi(y_1))(x^r - x_1^r) + \pi^2(y)y \log y) \ .$$

Here, in some occasions there may be possibilities of improving this simple and weak estimate, if we appeal to a rather deep result of D. A. Burgess on character sums (cf. D. A. Burgess: On character sums and L-series. II. Proc. London Math. Soc. (3) 13 (1963), 524-536).