数理解析研究所講究録
第 281 巻 1976 年 200-210

Note on Regular Fundamental Solutions and Some Other Topics

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In course of the preparation of the book [1], I have found some elementary but interesting results. Since I think it is not known in the literature, I would like to announce some of them.

§1. Singular spectrum of regular fundamental solutions.

Let $p(D)$ be a linear partial differential operator with constant coefficients. Regular fundamental solution means the following one defined by Hörmander:

$$(1.1)\quad E(x) = \frac{1}{(2\pi)^n} \sum_{\vartheta\in A} \int_{D_\vartheta} d\xi \frac{1}{2\pi} \oint_{|\tau|=1} \frac{e^{ix(\xi+\tau\vartheta)}}{p(\xi+\tau\vartheta)} \frac{d\tau}{\tau}.$$

Here A is a finite subset of R^n such that R^n is decomposed into closed subdomains D_ϑ, $\vartheta\in A$ on each of which holds

$$(1.2)\quad \sum_{|\alpha|\le m} |D^\alpha p(\xi)| \le C \inf_{|\tau|=1} |p(\xi+\tau\vartheta)|, \quad \text{if} \quad \xi\in D_\vartheta.$$

The integral in ξ converges as a distribution of x.

<u>Theorem 1.1</u> S.S.$E(x)$ is contained in $\{(x,\xi)\in R^n\times S^{n-1};\ x\xi = 0\}$.

This means that outside the origin $E(x)$ is analytic to the direction transversal to the sphere $|x| = \text{const}$.

<u>Proof</u> Fix a point x^0. We divide the integral of (1.1) into two parts. First consider the integral on the region

$|x^0\xi| \geq \delta|\xi|$. We show that this is analytic on a neighborhood of x^0. Note that there exists $c > 0$ independent of ξ such that

$$(1.3) \qquad \sum_{|\alpha|\leq m} |D^\alpha p(\xi)| \leq 2C \inf_{|\tau|=1} |p((1+\sigma)\xi+\tau\vartheta)|,$$
$$\text{if} \quad |\sigma| < c \quad \text{and} \quad \xi \in D_\vartheta.$$

In fact, by Taylor's theorem we have

$$|p((1+\sigma)\xi+\tau\vartheta) - p(\xi+\tau\vartheta)| = \Big|\sum_{0<|\alpha|\leq m} D^\alpha p(\xi+\tau\vartheta)(\sigma\xi)^\alpha\Big|$$
$$\leq |\sigma|C' \sum_{|\alpha|\leq m} |D^\alpha p(\xi)| .$$

Hence, taking $c = 1/2C'$ we obtain (1.3) due to (1.2). Thus employing the Abelian limit we can modify the path of integration from the real region $\pm x^0\xi \geq \delta|\xi|$ into $\zeta = \xi \pm ic\xi$ in the complex. We connect the modified path with the original one by a linear way on $\delta|\xi|/2 \leq |x^0\xi| \leq \delta|\xi|$. Consider, e.g., the integral

$$\lim_{\varepsilon\downarrow 0} \frac{1}{(2\pi)^n} \int_{D_\vartheta\cap\{x^0\xi\geq\delta|\xi|\}} d\zeta \frac{1}{2\pi} \oint \frac{e^{ix(\zeta+\tau\vartheta)-cx\xi-\varepsilon\zeta^2}}{p(\xi+\tau\vartheta)} \frac{d\tau}{\tau}.$$

If x belongs to the $\delta/2$-neighborhood of x^0, we have

$$-cx\xi \leq -cx^0\xi + c\delta|\xi|/2 \leq -c\delta|\xi|/2.$$

Hence, even after letting $\varepsilon = 0$, the above integral converges absolutely and can be continued analytically into the complex neighborhood $|\mathrm{Im}\, z| < c\delta/2$.

Next consider the original integral on the region $|x^0\xi| \leq \delta|\xi|/2$ and the modified one on $\delta|\xi|/2 \leq |x^0\xi| \leq \delta|\xi|$. We claim that their singular spectrum is contained in $R^n \times \{\xi\in S^{n-1}; |x^0\xi| \leq \delta\}$. Since $\delta > 0$ is arbitrary, this will prove the theorem. Divide $|x^0\xi| \leq \delta|\xi|/2$ into a finite number of proper convex closed cones Γ_j^o. It suffices to consider, e.g.,

$$(1.4) \qquad \int_{D_\vartheta \cap \Gamma_j^o} d\xi \int_{|\tau|=1} \frac{e^{ix(\xi+\tau\vartheta)}}{p(\xi+\tau\vartheta)} \frac{d\tau}{\tau}.$$

Replace x by the complex variable z. If the imaginary part is fixed in such a way as $\mathrm{Im}\, z \in \Gamma_j$, then we have $\mathrm{Im}\, z\xi \geq \varepsilon|\xi|$ with some $\varepsilon > 0$ for ξ on this region of integration. Therefore the integral converges absolutely and locally uniformly on the wedge $R^n + i\Gamma_j$, hence defines a holomorphic function there. We can easily see that the limit of the holomorphic function to the real domain agrees with the distribution (1.4). Thus we conclude that the singular spectrum of (1.4) is contained in $R^n \times \Gamma_j^o$. The argument is similar as to the modified integral on $\delta|\xi|/2 \leq |x^0\xi| \leq \delta|\xi|$, because the modification of the path does not increase the real part of the exponent. q.e.d.

As a corollary we can prove the Ehrenpreis-Komatsu existence theorem without employing functional analysis.

<u>Corollary 1.2</u> Let K be a convex compact set in R^n. Then $p(D) : A(K) \longrightarrow A(K)$ is surjective.

<u>Proof</u> Take $f \in A(K)$. Take a convex analytic hypersurface S surrounding K in the region where f is defined. Let $\chi(x)$ be the characteristic function of the interior of the surface. Then $S.S.\,\chi f$ is contained in the conormal bundle of S. Put

$$u(x) = E(x) * [\chi(x)f(x)] = \int E(x-y)\chi(y)f(y)dy.$$

By the estimation rule for singular spectrum on the operation of integration and product, we see that

$$S.S.u \subset \{(x+y,\xi) \in R^n \times S^{n-1};\ (x,\xi) \in S.S.E,\ (y,\xi) \in S.S.\chi f\}.$$

Hence the direction of the singular spectrum of u at a point $x^0 \in K$ comes from the point $y \in S$ where the line $\overline{x^0 y}$ is tangent to S. Since S is convex this is void. Thus u is a real analytic solution of $p(D)u = f$ on K. q.e.d.

Remark that the theory of (analytic) singular spectrum for distributions can be developped in a very elementary way employing the curved wave decomposition of the delta function. This is an idea due to M.Kashiwara. The singular spectral decomposition for f with compact support takes the form

$$(1.5)\qquad f(x) = \int_{S^{n-1}} W_{+0}(x,\omega) * f(x)d\omega,$$

$$W_{+0}(x,\omega) = \frac{(n-1)!}{(-2\pi i)^n} \frac{(1-ix\omega)^{n-1}-(1-ix\omega)^{n-2}(x^2-(x\omega)^2)}{(x\omega+i(x^2-(x\omega)^2)+i0)^n}.$$

This integral can be simply considered as the distribution limit of the Riemann sum

$$\sum_{k=1}^{N} W_{+0}(x,\omega^k) * f(x)\Delta\omega^k.$$

Especially the coherence of the concept of the singular spectrum for distributions and for hyperfunctions follows. In fact, the components of the decomposition (1.5) taken in the hyperfunction sense necessarily becomes distributions if f is a distribution. For further details see [1].

§2. General boundary value theory for distribution solutions.

Let $p(x,D)$ be an m-th order linear partial differential operator with infinitely differentiable coefficients. Assume that the hyperplane $x_1 = 0$ is non-characteristic with respect to p. Then for every distribution solution u of $p(x,D)u = 0$ on $x_1 > 0$ which is extensible as a distribution across $x_1 = 0$, we can define its boundary values to $x_1 = 0$. More precisely, let $U \subset R^n$ be a domain. Put

$$U^+ = U \cap \{x_1 > 0\}, \quad U^0 = U \cap \{x_1 = 0\}, \quad \overline{U^+} = U^+ \cup U^0.$$

A neighborhood of $\overline{U^+}$ or U^0 means an open set in R^n which contains $\overline{U^+}$ or U^0 as a closed subset. When we consider U^0 as an open subset of $x_1 = 0$, we let it be denoted by U', i.e., $U^0 = \{0\} \times U'$. Let $\{b_j(x,D)\}_{j=0}^{m-1}$ be a normal system of boundary operators, i.e., each $b_j(x,D)$ be a j-th order linear partial differential operator with infinitely differentiable coefficients with respect to which $x_1 = 0$ is non-characteristic. Let $\{c_j(x,D)\}_{j=0}^{m-1}$ be its dual system. It is defined by the formula

$$(2.1)\quad p(x,D)[\theta(x_1)u] = \sum_{j=0}^{m-1} {}^t c_{m-j-1}(x,D)[\delta(x_1) b_j(x,D)u] + \theta(x_1) p(x,D)u, \qquad u \in C^m(R^n).$$

If we assume that the coefficients of $c_j(x,D)$ do not contain x_1, the dual system is uniquely determined from p and b_j by this formula.

<u>Lemma 2.1</u> Let u be a distribution solution of $p(x,D)u = 0$ on U^+. Assume that u is extensible as a

distribution to a neighborhood of $\overline{U^+}$. Then there exist a unique extension $[u]_0^+ \in D'(U)$ of u satisfying $\operatorname{supp}[u]_0^+ \subset \overline{U^+}$ and unique data $u_j(x') \in D'(U')$ satisfying

$$(2.2) \quad p(x,D)[u]_0^+ = \sum_{j=0}^{m-1} {}^t c_{m-j-1}(x,D)[\delta(x_1)u_j(x')].$$

<u>Proof</u> By regularizing u locally and employing $\theta(x_1)$ and a partition of unity, we can anyway find an extension v of u satisfying $\operatorname{supp} v \subset \overline{U^+}$. Then $\operatorname{supp} p(x,D)v \subset U^0$. By the well known structure theorem, such a distribution is locally uniquely expressed in the form

$$p(x,D)v = \sum_{k=0}^{M} D_1^k \delta(x_1) f_k(x').$$

If $M \geq m$, employing the original equation we can write

$$D_1^M \underbrace{\delta(x_1)}_{} f_M(x') = p(x,D) D_1^{M-m}[\delta(x_1) f_M(x')/p_0(x)] + \sum_{k=0}^{M-1} D_1^k \delta(x_1) g_k(x'),$$

where $p_0(x)$ is the coefficient of D_1^m in p. Thus replacing v by $v - D_1^{M-m}[\delta(x_1) f_M(x')/p_0(x)]$, we can diminish M by one. Repeating this we can finally let $M = m-1$. The coefficients and the extension are then uniquely determined. In fact, assume that there are two such extensions $[u]_0^+$, $[u]_0^{+\prime}$. Then we have

$$p(x,D)([u]_0^+ - [u]_0^{+\prime}) = \sum_{k=0}^{m-1} D_1^k \delta(x_1) h_k(x'),$$

$$[u]_0^+ - [u]_0^{+\prime} = \sum_{j=0}^{N} D_1^j \delta(x_1) w_j(x'), \quad w_N(x') \not\equiv 0.$$

Substituting the latter into the former, we come to a contradiction. Reformulation to the case of general boundary system (2.2) is easy. q.e.d.

We define $[u]_0^+$ to be the canonical extension of u and $u_j(x')$ in (2.2) to be the boundary values of u with respect to the boundary system $\{b_j(x,D)\}$ and write

$$(2.3) \qquad b_j(x,D)u|_{x_1\to+0} = u_j(x'), \quad j = 0,\ldots,m-1.$$

If u is extensible as a function of class C^m or as a distribution solution of $p(x,D)u = 0$ to a neighborhood of $\overline{U^+}$, then the product $\theta(x_1)u$ is regitimate and by (2.1) we have

$$b_j(x,D)u|_{x_1\to+0} = b_j(x,D)u|_{x_1=0}.$$

Note that the right hand side of this equality does not have a meaning in general. The following lemma justifies the limit symbol of the boundary values.

Lemma 2.2 Let $U = \{|x_1|<\delta\}\times U'$. Let u be a distribution solution of $p(x,D)u = 0$ in U^+ which is extensible as a distribution to a neighborhood of $\overline{U^+}$. Then as $\varepsilon\downarrow 0$ we have

$$b_j(x,D)u|_{x_1=\varepsilon} \longrightarrow b_j(x,D)u|_{x_1\to+0} \quad \text{in } D'(U').$$

Proof It only suffices to consider the case $\{b_j(x,D)\} = \{D_1^j\}$. Since the convergence in D' is local we can replace U' by a smaller set and assume that $u = q(D)v$, where v is a function of class C^1 and $q(D)$ is an N-th order linear partial differential operator with constant coefficients with respect to which $x_1 = 0$ is non-characteristic. v is a solution of $p(x,D)q(D)v = 0$. The boundary values $v_j(x') = D_1^j v|_{x_1\to+0}$ of v with respect to this operator are given by

$$p(x,D)q(D)[\theta(x_1)v] = \sum_{j=0}^{m+N-1} r_{m+N-j-1}(D)[\delta(x_1)v_j(x')],$$

where $\{r_j\}$ is the dual system of $\{D_1^j\}_{j=0}^{m+N-1}$ with respect to this operator. In fact, the left hand side is obviously of order at most m+N-1 as a distribution, hence the higher order terms do not appear. Letting $\varepsilon \downarrow 0$, we have $\theta(x_1-\varepsilon)v \longrightarrow \theta(x_1)v$ in $D'(U)$. Hence

$$\sum_{j=0}^{m+N-1} r_{m+N-j-1}(D)[\delta(x_1-\varepsilon)D_1^j v|_{x_1=\varepsilon}] \longrightarrow \sum_{j=0}^{m+N-1} r_{m+N-j-1}(D)[\delta(x_1)v_j(x')].$$

Multiplying by $(ix_1)^j/j!$, $j = m+N-1,\dots,0$ and taking the definite integral with respect to x_1, we see that the assertion of the lemma is true for the solution v. Now we have

$$\begin{aligned}\theta(x_1-\varepsilon)u &= \theta(x_1-\varepsilon)q(D)v \\ &= q(D)[\theta(x_1-\varepsilon)v] + \sum_{j=0}^{N-1} q_{N-j-1}(D)[\delta(x_1)D_1^j v|_{x_1=\varepsilon}].\end{aligned}$$

Each term of the right hand side converges in $D'(U)$ due to the above argument. Thus setting $[u] = \lim_{\varepsilon\downarrow 0} \theta(x_1-\varepsilon)u$ we have

$$\begin{aligned}p(x,D)[u] &= \lim_{\varepsilon\downarrow 0} p(x,D)(\theta(x_1-\varepsilon)u) \\ &= \lim_{\varepsilon\downarrow 0} \sum_{j=0}^{m-1} p_{m-j-1}(x,D)[\delta(x_1)D_1^j u|_{x_1=\varepsilon}].\end{aligned}$$

Employing $(ix_1)^j/j!$ as above, we conclude that $[u]$ is the canonical extension of u and each $D_1^j u|_{x_1=\varepsilon}$ converges to the corresponding boundary value. q.e.d.

In the same way, for a distribution solution u on $U^- = U \cap \{x_1<0\}$ which is extensible as a distribution to a neighborhood of $\overline{U^-} = U^- \cup U^0$, we can define the canonical extention $[u]_0^-$ and the boundary values $u_j(x') = b_j(x,D)u|_{x_1\to -0}$ by the formula

$$(2.4) \qquad -p(x,D)[u]_0^- = \sum_{j=0}^{m-1} {}^t c_{m-j-1}(x,D)[\delta(x_1)u_j(x')].$$

As for the uniqueness of the boundary value problem we have

<u>Theorem 2.3</u> Let $u^{\pm}$ be distribution solutions of $p(x,D)u = 0$ defined resp. on $U^{\pm}$ and extensible as distributions to a neighborhood of $\overline{U^{\pm}}$. Assume that

$$b_j(x,D)u^{+}|_{x_1 \to +0} = b_j(x,D)u^{-}|_{x_1 \to -0}, \quad j=0,\dots,m-1.$$

Then there exists a unique distribution solution u on U satisfying $u|_{U^{\pm}} = u^{\pm}$ and

$$b_j(x,D)u^{+}|_{x_1 \to +0} = b_j(x,D)u^{-}|_{x_1 \to -0} = b_j(x,D)u|_{x_1=0}.$$

<u>Proof</u> Let $[u^{\pm}]_0^{\pm}$ be the canonical extensions of $u^{\pm}$. Then $u = [u^{+}]_0^{+} + [u^{-}]_0^{-}$ satisfies $p(x,D)u = 0$ on U. The uniqueness is deduced in the same way as in the proof of Lemma 2.1. q.e.d.

<u>Corollary 2.4</u> Assume that $p(x,D)$ is obtained from an operator with analytic coefficients by an infinitely differentiable coordinate transformation. Then the boundary values (2.3) determines the solution u on U^{+} locally uniquely on a neighborhood of $x_1 = 0$.

<u>Proof</u> Consider the difference v of two solutions with the same boundary data (2.3). Then we have $p(x,D)[v]_0^{+} = 0$. Therefore by the Holmgren theorem we conclude that $[v]_0^{+} \equiv 0$, hence $v \equiv 0$ on a neighborhood of $x_1 = 0$. q.e.d.

As for the solvability of the boundary value problem we have

<u>Theorem 2.5</u> Let $E(x,y)$ be a fundamental solution of $p(x,D)$, i.e., $p(x,D)E(x,y) = \delta(x-y)$. Then we have for $y \in \{x_1=0\}$ fixed,

$$b_{m-1}(x,D)E|_{x_1 \to +0} - b_{m-1}(x,D)E|_{x_1 \to -0} = \frac{ib^0_{m-1}(0,x',\nu)}{p^0(0,x',\nu)}\delta(x'-y'),$$

$$b_j(x,D)E|_{x_1 \to +0} - b_j(x,D)E|_{x_1 \to -0} = 0, \quad 0 \leq j \leq m-2.$$

Here p^0 or b^0_{m-1} denotes the principal part of p or b_{m-1} and $\nu = (1,0,\ldots,0)$.

Proof Setting $b_j(x,D)E|_{x_1 \to \pm 0} = u^{\pm}_j(x')$, $j = 0,\ldots,m-1$, we have

$$p(x,D)[E]^{\pm}_0 = \pm \sum_{j=0}^{m-1} {}^t c_{m-j-1}(x,D)[\delta(x_1)\ u^{\pm}_j(x')].$$

Hence

$$p(x,D)([E]^+_0 + [E]^-_0) = \sum_{j=0}^{m-1} {}^t c_{m-j-1}(x,D)[\delta(x_1)(u^+_j(x')-u^-_j(x'))].$$

On the other hand, we know that

$$p(x,D)E = \delta(x-y) = {}^t c_0(x,D)\left[\delta(x_1)\ \frac{ib^0_{m-1}(0,x',\nu)}{p^0(0,x',\nu)}\ \delta(x'-y')\right]$$

Make the difference of these two equalities. Then due to the uniqueness we conclude that $E = [E]^+_0 + [E]^-_0$ and the theorem follows. q.e.d.

Employing the fundamental solution in a standard way, we can locally solve the bilateral boundary value problem.

Corollary 2.6 Let $u_j(x')$, $j = 0,\ldots,m-1$ be distributions whose supports are contained in the region where the fundamental solution of $p(x,D)$ is defined. Then we can locally find solutions $u^{\pm}$ of $p(x,D)u = 0$ on $\pm x_1 > 0$ satisfying

$$b_j(x,D)u^+|_{x_1 \to +0} - b_j(x,D)u^-|_{x_1 \to -0} = u_j(x'),\ j=0,\ldots,m-1.$$

The pair $u^{\pm}$ is locally uniquely determined up to those which can be connected as a solution on a neighborhood of $x_1 = 0$.

Proof Taking a suitable derivative $p_k(y,D_y)E(x,y)|_{y_1=0}$ by an operator of order k, we can find a solution $E_k(x,y')$ of $p(x,D)u = 0$ on $R^n \setminus (0,y')$ satisfying

$$b_j(x,D)E_k(x,y')|_{x_1\to +0} - b_j(x,D)E_k(x,y')|_{x_1\to -0} = \delta_{j,m-k-1}.$$

Then

$$u^{\pm}(x) = \sum_{k=0}^{m-1} \int E_{m-k-1}(x,y')u_k(y')dy', \quad \text{on} \quad \pm x_1 > 0,$$

is the desired solution. q.e.d.

Since the calculation of this section is purely algebraic, we can perform the same argument to the operator of the type :

$$p(x,D) = D_1^m + \sum_{k=0}^{m-1} p_{m-k}(x,D')D_1^k$$

where the order of $p_k(x,D')$ may be greater than k. Thus, for the distribution solution we can give a local definition of the boundary value, hence the restriction to a kind of characteristic hypersurface.

Reference

[1] Kaneko, A., Teisû Keisû Senkei Henbibun Hôteishiki, Iwanami Kôza Kiso Sûgaku, to appear.