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The Firing Squad Synchronization Problem for Graphs

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Abstract This paper deals with the Firing Squad Synchronization problem for some classes of digraph structures and graph structures. The first part of this paper gives solutions for the classes of circuit structures, quasi-circuit structures, and some other extended digraph structures. The second part gives a solution for the class of connected graph structures, whose synchronization time for a graph structure with radius r is $3r+1$ or $3r$ time units.

1. Introduction

The problem of synchronizing a finite (but arbitrarily long) one-dimensional array of finite automata, known as the firing squad synchronization problem, was proposed by Myhill and Moore [8]. Consider a one-dimensional array of identical finite automata. The state of each automaton at time $t+1$

depends on its own state and those of its two neighbours at time t. The problem consists of defining the structure of automata so that one end automaton of the array, called the general, can cause all automata to enter a particular state, called the firing state, all at once.

It can easily be shown that the minimal time required to synchronize an n-element array is $2n-2$ time units. The first minimal-time solution was obtained by Goto [2]. Waksman [13] has produced a minimal-time solution with 16 states and Balzer [1] has reduced the complexity to 8 states. The problem was generalized in many different ways by Moore and Langdon [9], Herman [3, 4], Rosenstiehl [11,12], Kobayashi [5, 6, 7], and Romani [10].

This paper deals with the firing squad synchronization problem for d-digraph structures and d-graph structures. Informally, a d-digraph structure (d-graph structure) is a network of identical finite automata in which an automaton is placed at each vertex of a digraph (graph) and the automata are connected along every arcs (edges) of the digraph (graph).

We present solutions of the problem for some subclasses of d-digraph structures in section 3 and for the class Π^d of connected d-graph structures in sections 4 and 5.

Rosenstiehl and Romani studied the problem of synchronizing a network of finite automata however connected. Rosenstiehl's solution obtains a synchronization time of $2n$, where n is the number of automata in the network, and Romani's solution obtains a synchronization time shorter than or equal to that of Rosenstiehl's. The class of networks studied by Rosenstiehl and Romani is the same class as Π^d in our formulation. Our solution obtains a synchronization time of $3r$ or $3r+1$, where r is the longest distance between the general and any other element in the network.

2. Preliminaries

In this section, we give definitions and notations used in this paper.

A digraph (or directed graph) is a pair (X, U), where X is a set of elements called vertices and U is a set of ordered pairs of distnct vertices called arcs.

A graph (or undirected graph) G is a pair (X, E) where X is a set of vertices and E is a set of unordered pairs of distinct vertices called edges. A graph G is also regarded as a symmetric digraph G^* that has two oppositely directed arcs corresponding to each edge of G. In this paper we adopt this viewpoint. The order of a digraph (graph) G, denoted by $|G|$, is the number of vertices in G.

The distance from a vertex x to a vertex y in G, denoted by $\text{dist}_G(x, y)$, is the shortest length among the pathes from x to y. Note that generally $\text{dist}_G(x, y) \neq \text{dist}_G(y, x)$ in a digraph G.

A d-finite automaton M^d is a 6-tuple $(S, s_e, s_q, s_g, s_f, \lambda)$, where (1) S is a finite set of states, (2) s_e is an element not in S (the external signal), (3) s_g, s_q, and s_f are particular distinct elements in S (the quiescent state, the general state, and the firing state respectively), and (4) λ is a transition function from $S \times (S \cup \{s_e\})^d$ into S such that λ $(s_q, s_1, \cdots, s_d) = s_q$ if each of $s_1, \cdots, s_d$ is either s_q or s_e. Informally, M^d is an automaton with d input terminal. A d-tuple $(s_1, \cdots, s_d)$ in the set $(S \cup \{s_e\})^d$ is called an input letter.

A d-digraph structures is a 3-tuple (G, x_g, d), where d is a positive integer, G is a digraph such that $d_G^- \leq d$ where d_G^- is the in-degree of G, and x_g is a particular vertex of G called the general. On a d-digraph structure, a d-finite automaton M^d is placed at each vertex of G. A vertex x installed with a d-finite automaton is called a cell x. Cells are connected along arcs in G. Let x be a vertex with $d_G^-(x)$ into-arcs. Among d input terminals of a cell x, $d_G^-(x)$ of them are connected with the output terminals of the predecessor cells of x and the remaining $d - d_G^-(x)$ (≥ 0) input terminals are connected to the external world.

In order to describe clearly how the input terminals of x are connected to the predecessors of x or the external world, each input terminal is labeled a distinct integer i $(1 \leq i \leq d)$. If the input terminal of x, labeled i, is connected to a predecessor y of x, y is called the i th predecessor cell of x. If the input terminal labeled i is connected to the external world, we say that the i th predecessor does not exist. The i-th component of an input letter $(s_1, \cdots, s_d)$ of a cell x is the state of the i th predecessor cell of x (if it exists) or the external signal s_e (if not).

A d-digraph structure (G, x_g, d) is called a connected d-digraph structure if there is at least one path from x_g to y for any vertex y in G. In the followings, we shall deal with connected digraph structures, so we call them simply digraph structures.

Suppose that a d-digraph structure (G, x_g, d) is given and a d-finite automaton M^d is placed on each vertex of G. Then the state of a cell x at time t, denoted by $s\ (x, t, G, x_g, M^d)$, is defined by the following rules.

At $t = 0$, only the general cell x_g is in the general state s_g and all other cells are in the quiescent state s_q. That is, $s\ (x, 0, G, x_g, M^d)$ is s_g if $x = x_g$ and is s_q otherwise. Let

$$s_i = s\ (x_i, t, G, x_g, M^d) \quad \text{if the } i \text{ th predecessor } x_i \text{ of } x \text{ exists}$$

$$= s_e \quad \text{if it does not exist.}$$

Then the state of x at time $t + 1$ is determined as

$$s\ (x, t + 1, G, x_g, M^d) = \lambda\ (s(x, t, G, x_g, M^d), s_1, \cdots, s_d).$$

If the state of x at time t is s_f, we say that x fires at time t. The problem is to specify a automaton M^d which makes all cells in (G, x_g, d) to fire at once. A d-finite automaton M^d is called a solution of the firing squad synchronization problem for a subclass Θ^d of d-digraph structures (simply a solution for Θ^d) if, for each d-digraph structures (G, x_g, d) in Θ^d, there exists a time $t\ (G, x_g, d, M^d)$ such that all cells in G fire at time $t\ (G, x_g, d, M^d)$ and do not fire prior to time $t\ (G, x_g, d, M^d)$. The time $t\ (G, x_g, d, M^d)$ is called the synchronization time of M^d for firing a d-digraph structures $(G, x_g, d) \in \Theta^d$, (simply the synchronization time of M^d for (G, x_g, d)).

Next, we define a d-graph structure(G,x_g,d). Let G be a graph with $d_G \leq d$, and let x_g be a particular vertex of G. For G, we define the symmetric digraph G^* that has two oppositely directed arcs corresponding to each edge in G. Thus $d_G = d^-_{G^*} = d^+_{G^*}$. Then a d-graph structure (G, x_g, d) is defined to be the d-digraph structure (G^*, x_g, d).

A d-graph structure (G, x_g, d) is called a connected d-graph structure if there is at least one path from x to y for any pair of distinct vertices. The class of connected d-graph structures and the corresponding d-digraph structures are denoted as Π^d and Π^{d*} respectively. In the followings, connected graph structures are called simply graph structures.

A d-finite automaton M^d is called a solution of the firing squad synchronization problem for Π^d if M^d is a solution for Π^{d*}. The synchronization time $t(G, x_g, d, M^d)$ of M^d for a d-graph structure (G, x_g, d) is defined by the synchronization time $t(G^*, x_g, d, M^d)$ for the corresponding d-digraph structure (G^*,x_g, d). For a d-graph structures (G, x_g, d), let $t_{min}(G, x_g, d)$ be the minimum value of $t(G, x_g, d, M^d)$ over all solutions M^d for Π^d. Given (G, x_g, d), let $L(G, x_g, d)$ be $\max_{x,y} \{dist_G(x_g, x) + dist_G(x, y)\}$. Kobayashi gave the following result about $t_{min}(G, x_y, d)$ [5]

Theorem 2.1. For $(G, x_g, d) \in \Pi^d$ with $|G| = 1$,

$$t_{min}(G, x_g, d) = 1.$$

For $(G, x_g, d) \in \Pi^d$ with $|G| \geq 2$,

$$t_{min}(G, x_g, d) \geq L(G, x_g, d).$$

Especially, if there are three cells x, x', and y such that x and x' are adjacent, $dist_G(x_g, x) = dist_G(x_g, x')$, and $dist_G(x_g, x) + dist_G(x, y) = dist_G(x_g, x') + dist_G(x', y) = L(G, x_g, d)$, then

$$t_{min}(G, x_g, d) \geq L(G, x_g, d) + 1.$$

Intuitively, $L(G, x_g, d)$ is the time required for x to receive a signal from x_g and leave the quiescent state, and then for y to receive a signal from x for any vertices x and y in G.

3. Solutions for certain subclasses of d-digraph structures.

3.1 In this section, we give solutions for certain subclasses of d-digraph structures. A digraph $C_n = (X_n, U_n)$ is called a circuit if $X_n = \{x_0, \cdots, x_{n-1}\}$, $U_n = \{u_0, \cdots, u_{n-1}\}$, $u_i = (x_{i-1}, x_i)$ for each i $(0 < i < n)$, and $u_0 = (x_{n-1}, x_0)$. A circuit structure $(C_n, x_0, 1)$ is a 1-digraph structure in which C_n is a circuit. Let Θ_c be the class of circuit structures.

A solution for Θ_c was given by Kobayashi. Its synchronization time for $(C_n, x_0, 1)$ is 2n-1 time units. It is easily shown that the minimum time required to synchronize $(C_n, x_0, 1)$ is 2n-1 time units. So the solution given by Kobayashi is a minimum time solution. The authors have obtained independently a similar solution. Here we give our solution $M_c = (S_c, s_e, s_q, s_g, s_f, \lambda_c)$ which is called the circuit solution.

The evolution of the solution M_c is depicted in Fig. 1. The horizontal axis represents the circuit of cells in C_n and the vertical axis represents time. The (z, t) entry represents the state of the z th cell at time t.

Let us divide cells in C_n into two equal parts. We shall represent a binary number $n = a_0 + a_1 2 + \cdots + a_m 2^m$ ($a_i = 0$ or 1) as $n = \langle a_1, \cdots, a_m \rangle$. In dividing n cells into two equal parts, each part is considered to contain $\langle a_1, \cdots, a_m \rangle$ cells. We divide the two halves into two parts each so that the size of each subdivision is $\langle a_2, \cdots, a_m \rangle$. In similar fashion, the size of the k th subdivision is $\langle a_k, \cdots, a_m \rangle$.

We use four signals P_{00}, P_{11}, P_{20}, and P_{21} for marking the boundaries between subdivisions and also for generating the following series of signals which propagate along the circuit. P_{00} and P_{11} are called general signals, and P_{20} and P_{21} are called subgeneral signals.

A general signal P_{00} generates following series of signals:
a P-series consisting of P_0 and P_1 signals which does not propagate,
BC-series consisting of B_0, B_1, B_2, B_3, C_0, C_1, C_2, and C_3 signals which propagate with velocities $v = 1/3, 3/7, \cdots, (2^i-1)/(2^{i+1}-1), \cdots$ (cells/time unit), (a BC-series which propagates with $v = (2^i-1)/(2^{i+1}-1)$ is called a $(BC)_i$-series),
an A_0-series consisting of A_{00} and A_{01} signals which propagates with $v = 1$, and
RS-series consisting of R_1, R_2, S_0, S_1, and S_2 signals which propagate with $v = 2/3, 4/7, \cdots, 2^i/(2^{i+1}-1), \cdots (i = 1, 2, \cdots)$, (an RS-series which propagates with $v = 2^i/(2^{i+1}-1)$ is called an $(RS)_i$-series).

A general signal P_{11} generates following series of signals:
a P-series,
BC-series which propagate with the same velocities as those of the above BC-series but are delayed one time unit,
an A_1-series consisting of A_{10} and A_{11} signals which propagates with $v = 1$, and RS-series.

A subgeneral signal P_{2l} ($l = 0$ or 1) at (z, t) generates P_{ll} signal at $(z+1, t+1)$ and a P_2-series consisting of P_2 signals which does not propagate.

A $(BC)_i$-series is obtained if we delay a series, which propagates with $v = 1/2$, one unit time on every $2^{i-1}-1$ cells. It is shown that a $(BC)_{i+1}$-series is produced by a $(BC)_i$-series inductively.
A $(RS)_i$-series is obtained if we advance a series, which propagates with $v = 1/2$, one unit time on every 2^i cells.
It is shown that a $(RS)_{i+1}$ series is produced by a $(RS)_i$-series inductively.
We shall show how general and subgeneral signals and are generated on boundaries of subdivision. General and subgeneral signals are generated according to the following rules.

(1) When an A_0-series meets C_2 of a BC-series, P_{00} is generated.

(2) When an A_0-series meets B_2 of a BC-series, P_{11} is generated.

(3) When an A_1-series meets C_3 or C_0 of a BC-series, P_{20} is generated.

(4) When an A_1-series meets B_3 or B_0 of a BC-series, P_{21}

is generated.

(5) When a P-series meets S_2 or S_0 of an RS-series, P_{00} is generated.

(6) When a P-series meets R_2 of an RS-series, P_{11} is generated.

(7) When a P_2-series meets S_2 or S_0 of an RS-series, P_{20} is generated.

(8) When a P_2-series meets R_2 of an RS-series, P_{21} is generated.

(9) When a P-series meets A_{0*} of an A_0-series, P_{**} is generated (* = 0 or 1).

Four cases are to be considered.

Case 1. let n be an integer represented by $<a_0, \cdots, a_m>$, and suppose that P_{00} is generated at (0, 0) and $P_{a_0a_0}$ is generated at (n, n). It is shown that $P_{a_ia_i}$ is generated at $(n, 2n- <a_i, \cdots, a_m>)$ (i= 1, 2, $\cdots$).

Case 2. Suppose that P_{00} is generated at (0, 0) and $P_{a_0a_0}$ is generated at $(0, n)$. It is shown that $P_{a_ia_i}$ is generated at $(n - <a_i, \cdots, a_m>, 2n - <a_i, \cdots, a_m>)$, and if $a_{i-1} = 1$, P_{2a_i} is generated at $(n - <a_i, \cdots, a_m> - 1, 2n - <a_i, \cdots, a_m> - 1)$.

Case 3. Suppose that P_{11} is generated at (0, 0), P_{2a_0} is generated at (n, n), and $P_{a_0a_0}$ is generated at $(n+1, n+1)$. It is shown that P_{2a_i} is generated at $(n, <a_i, \ldots, a_m>)$ by the similar way as in case 1 and $P_{a_ia_i}$ is generated at $(n+1, 2n - <a_i, \cdots, a_m> + 1)$.

Case 4. Suppose that P_{11} is generated at (0, 0) and

$P_{a_0 a_0}$ is generated at $(0, n+1)$. By the similar way as in case 2 and by considering that the BC-series generated by P_{11} propagate with one time unit delay, it is shown that $P_{a_i a_i}$ is generated at $(n - \langle a_i, \cdots, a_m\rangle, 2n - \langle a_i, \cdots, a_m\rangle + 1)$ and if $a_{i-1} = 1$, P_{2a_i} is generated at $(n - \langle a_i, \cdots, a_m\rangle - 1, 2n - \langle a_i, \cdots, a_m\rangle)$.

From the above consideration, we conclude that the general or subgeneral signals are generated synchronously at boundaries of subdivisions. Then, it is seen that all cells fire at time $(2n-1)$ for $(c_n, x_0, 1)$ and thus M_c is a solution for Θ_c.

Theorem 3.1. $M_c = (S_c, s_e, s_g, s_q, s_f, \lambda_c)$ is a solution for the class Θ_c of circuit structures and its synchronization time for $(c_n, x_0, 1)$ is $2n-1$ time units. The number of states of M_c is 38.

The evolution of the circuit solution M_c for $(c_{13}, x_0, 1)$ is given in Fig. 2, where s_g, s_f, and s_q are denoted by P_{00}, F, and blank respectively.

3.2. We consider a digraph $C_n^1 = (X_n^1, U_n^1)$ which is called a quasi-circuit. A quasi-circuit $C_n^1 = (X_n^1, U_n^1)$ is defined as follows.

(1) $X_n' = \{x_{ij} \mid 0 \le i \le n-1,\ 0 \le j \le h_i,\ h_0 = 0\}$.

(2) $U_n' = \{(x_{i-1\,0}, x_{i0}) \mid 0 \le i \le n-1,\ x_{-1\,0} = x_{n-1\,0}\}$

$$\bigcup_{0 \le i \le n-1} \ \bigcup_{0 \le i \le h_j} U_{ij},$$

where U_{ij} is the set of arcs of the form $(x_{i-1\,k}, x_{ij})$ for some k.

(3) There exists at least one path from x_{00} to x_{ij} for all vertices x_{ij}.

From the definition, it is shown that

(1) $dist_{C_n^1}(x_{00}, x_{ij}) = i < n$,

(2) there exists at least one circuit in C_n^1, and

(3) all circuits of C_n^1 pass through x_{00} and their length are n.

A d-quasi-circuit structure is a d-digraph structure (C_n^1, x_{00}, d) where C_n^1 is a quasi-circuit. Let Θ_{c1}^d be the class of d-quasi-circuit structures. A solution for Θ_{c1}^d can be obtained by slightly modifying the circuit solution M_c. Let $M_{c1}^d = (S_c, s_e, s_g, s_q, s_f, \lambda_{c1}^d)$ be a d-finite automaton whose input letters are d-tuples. The state transition function λ_{c1}^d is defined only for such input letters that all components other than s_e are identical signals. For these defined inputs, M_{c1}^d behaves as M_c does. In more detail, let an input letter of M_{c1}^d whose every component is either $s \in S_c$ or s_e be expressed as s^d. We define $\lambda_{c1}^d(s', s^d) = \lambda_c(s', s)$ for all $s', s \in S_c$ where λ_c is the state transition function of M_c.

Theorem 3.2. M_{c1}^d is a solution for Θ_{c1}^d and its synchronization time for $(C_n^1, x_{00}, d) \in \Theta_{c1}^d$ is $2n-1$ time units.

Proof. It is easily proved by the induction on t that at any time t and for each i $(0 \leq i \leq n-1)$, the state of the automaton at x_{ij} is independent of j, that is, $s(x_{ij}, t, C_n^1, x_{00}, M_{c1}^d)$ is identical for all $j (0 \leq j \leq h_i)$.

Since there is at least one circuit in (C_n^1, x_{00}, d), all cells on the circuit fire at time $2n-1$. Hence all cells in

(C_n^1, x_{00}, d) fire at time $2n-1$. M_{c1}^d is called the quasi-circuit solution.

3.3 We consider a digraph $C_n^2 = (X_n^2, U_n^2)$ which is defined as follows.

(1) There is at least one circuit in C_n^2 and all circuits in C_n^2 pass through a designated vertex x_{00}.

(2) The maximum length of circuits in C_n^2 is n.

(3) For each vertex x, there is at least one path from x_{00} to x and the maximum length of paths, in which no vertex is encountered more than once, is less than n.

In other words, C_n^2 is obtained by adding arcs of the form $(x_{ij}, x_{i'j'})$ with $i < i'-1$ to a quasi-circuit C_n^1.

Let Θ_{c2}^d be the class of d-digraph structures (C_n^2, x_{00}, d) and M_{c2}^d be the d-finite automaton which is obtained by modifying the quasi-circuit solution M_{c1}^d as explained below. M_{c2}^d consists of M_{c1}^d and the processer for the input signal. The processor finds which predecessor cells move to non-quiecent state in M_{c1}^d lastly. Since then, the processor regards the signals from the predecessors other than the lastly activated ones as external signals. In other words, M_{c2}^d disregards input signals received through arcs $(x_{ij}, x_{i'j'})$ with $i < i'-1$ stated for defining C_n^2 from C_n^1. Then Theorem 3.3 is easily proved

Theorem 3.3 M_{c2}^d is a solution for Θ_{c2}^d and its synchronization time for $(C_n^2, x_{00}, d) \in \Theta_{c2}^d$ is $2n-1$ time units.

3.4 We consider a digraph $C_n^3 = (X_n^3, U_n^3)$ which is defined as follows.

(1) There exists at least one circuit which pass through a

designated vertex x_{00}.

(2) The minimum length of circuits passing through x_{00} is n.

(3) For each x, there exists at least one path from x_{00} to x and $dist_{C_n^3}(x_{00}, x) < n$.

Let Θ_{c}^{d} be the class of d-digraph structures (C_n^3, x_{00}, d) and M_{c3}^{d} be the d-finite automaton which consists of the quasi-circuit solution M_{c1}^{d} and the processor for the input signal. The processor finds which predecessor cells move to non-quiecent states first. Since then, the processor regards the signals from the predecessors other than the first activated ones as external signals. Then Theorem 3.4 is easily proved.

Theorem 3.4 M_{c3}^{d} is a solution for Θ_{c3}^{d} and its synchronization time for (C_n^3, x_{00}, d) is $2n-1$ time units.

4. Two preliminary solutions for d-graph structures.

4.1. In this section, we shall consider the class Π^d of d-graph structures, and give two preliminary solutions M_{4r}^{d} and M_{3r+1}^{d} for Π^d. Let (G_r, x_g, d) be a d-graph structure with the radius r. Here, the radius r of a graph structure (G, x_g, d) is defined by $r = \max_{x \in G} dist_G(x_g, x)$. It will be shown that the synchronization time of M_{3r+1}^{d} for (G_r, x_g, d) is $3r+1$ time units. We call M_{3r+1}^{d} a $3r+1$ solution.

Before explaining the essential idea for constructing M_{3r+1}^{d}, we shall give a preliminary solution M_{4r}^{d} whose synchronization time for (G_r, x_g, d) is $4r$ time units. We call M_{4r}^{d} a $4r$ solution. The principal idea is to construct the automaton which reduces a given d-graph structure to a d-quasi-circuit structures and then simulate the quasi-circuit solution M^d_{ci}.

In $(G, x_g, d) \in \Pi^d$, if a cell x has no adjacent cell y such that $dist_G(x_g, y) > dist_G(x_G, x)$, then x is called a terminal cell. For each cell x, there is at least one path $\mu = [x_0\ x_1 \cdots x_l]$ such that $x_0 = x$, x_l is a terminal cell, and for all j $(0 \le j < l)$, $dist_G(x_g, x_j) < dist_G(x_g, x_{j+1})$. When x_1 is the i th adjacent cell of $x = x_0$, the path μ is called the i th path of x. The maximum length of the i th paths of x is denoted by $m(x, i)$. Note that if x is a terminal cell, $m(x, i)$ is 0 for all i.

A d-graph structure $(G_r, x_g, d) \in \Pi^d$ is reduced to a d-quasi-circuit structure $(C^1_{2r}, x_g, d) \in \Theta^d_{ci}$ as follows. (See Fig. 3.)

First, we remove every edge $e = [x, y]$ in G_r such that $dist_{G_r}(x_g, x) = dist_{G_r}(x_g, y)$ and obtain an d-graph G'_r. Then we divide each cell x other than the general cell x_g and terminal cells into two subcells x^1 and x^2 called the first subcell and the second subcell respectively and replace each edge $e = [x, y]$ in G'_r, for which $dist_{G'_r}(x_g, x) < dist_{G'_r}(x_g, y)$, with two arcs $u^1 = (x^1, y^1)$ and $u^2 = (y^2, x^2)$. (For the general cell (terminal cells), $x^1(y^1) = x^2(y^2) = x\ (y)$.) Finally, for each x in G'_r and i $(1 \le i \le d)$, if there exists j such that $m(x, i) < m(x, j)$, then we remove (x^2_i, x) where x^2_i is the second subcell of the i th adjacent cell x_i of x. Then we obtain a quasi-circuit C^1_{2r}.

The solution $M^d_{4r} = (S_{4r}, s_e, s_g, s_q, s_f, \lambda_{4r})$ first simulates the above reducing process. Its state set S_{4r} is given by $S_1 \times S_2 \times S_3 \times S_4 \cup \{s_f\}$. s_f is the firing state of M^d_{4r}. S_1 and S_2 are used to simulate the reducing process. S_3 and S_4 are used to simulate M^d_{ci} on the d-quasi-circuit structure reduced from a given d-graph structure.

We put

$$S_1 = \{G_0, G_1, G_2, H_0, H_1, I, J, Q_0\},$$
$$S_2 = \{0, 1, 2, 3\}^d,$$
$$S_3 = S_4 = S_c - \{F\},$$

where S_c is the state set of the quasi-circuit solution M^d_{ci} and F is its firing state.

The general cell starts from G_0 and reaches to G_2 via G_1 and each terminal cell starts from Q_0 and reaches to I via H_0. Each non-terminal cell starts from Q_0 and reaches to J through H_0 and H_1.

Each element of S_2 is expressed as $(m_1, \cdots, m_i, \cdots, m_d)$ where $m_i \in \{0, 1, 2, 3\} (1 \le i \le d)$. Let x be a cell and x_i be the i th adjacent cell of x. Let the S_2 component of x is $(m_1, \cdots, m_d)$. The value of m_i has the following meaning. $m_i = 1$ means that $dist_G(x_g, x) < dist_G(x_g, x_i)$ and thus the arcs (x_i^1, x^1) exists in C^1_{2r}. $m_2 = 2$ means that $dist_G(x_g, x) < dist(x_g, x_i)$ and thus the arcs (x^1, x_i^1) and (x_i^2, x^2) exist in C^1_{2r}. $m_i = 3$ means either that $dist_G(x_g, x) = dist_G(x_g, x_i)$ or x_i does not exist and thus the edge (x, x_i) is removed from G_r, or that there exists i such that $m(x, j) > m(x, i)$ and thus the arc (x_i^2, x^2) is removed from G_1. $m_i = 0$ means that the connection between x and x_i is not yet determined.

Initially, the general cell is in the state $(G_0, (0, \cdots, 0))$ and other cells are in $(Q_0, (0, \cdots, 0))$.

In simulating the behaviors of M^d_{c1}, M^d_{4r} makes each subcell to receive input signals only from its predecessor subcells in C^1_{2r} and ignore those from other subcells.

It is shown that M^d_{4r} can simulate the reduction from G_r to C^1_{2r} and the solution M^d_{c1} on the d-quasi-circuit structure (C^1_{2r}, x_g, d). Fig. 3 illustrates the solution M^d_{4r} on a $(G_3, x_g, 3)$.

Let $dist_G\ (x_g, x)$ and $\max_i m(x, i)$ be denoted by l^1_x and l^2_x. It is seen that the arcs incident into x^1 are established at $t = l^1_x$ and arcs incident into x^2 are established at $t = l^1_x + 2l^2_x + 1$. Hence, the general cell can start to simulate M^d_{c1} at $t = 1$.

We define that each cell moves to s_f when its two subcells move to F. Then the synchronization time of M^d_{4r} for (G_r, x_g, d) is $1 + (2(2r) - 1) = 4r$ time units.

Theorem 4.1 M^d_{4r} is a solution for Π^d and its synchronization time for (G_r, x_g, d) is $4r$ time units if $|G_r| \geq 2$ and 1 if $|G| = 1$.

Next, we shall describe the $3r + 1$ solution M^d_{3r+1}. M^d_{3r+1} simulates the reduction from (G_r, x_g, d) to (C^1_{2r}, x_g, d) as M^d_{4r} does. Kobayashi pointed out the following two facts about (C^1_{2r}, x_g, d) and suggested that the synchronization time for (G_r, x_g, d) is improved to about $3r$ time units.

(1) Synchronization of (G_r, x_g, d) is achieved by synchonizing the subdigraph structure of (C^1_{2r}, x_g, d) consisting

only of the first cells.

(2) A terminal cell x_M farthest from x_g devides a circuit in (C^1_{2r}, x_g, d) into two halves. x_M is called the center of C^1_{2r}. We can find the center of C^1_{2r} at time r in the reduction process.

Considering the two facts, we have the following modified problem. Let $(C_{2n}, x_0, x_n, 1)$ be a circuit structure in which the center x_n of C_{2n} is designated. Let $(x_0, x_1, \cdots, x_n, \cdots, x_{2n-1})$ be the circuit. Find a solution of synchronizing all cells on the semicircuit $(x_0, x_1, \ldots, x_n)$ for the class of $(C_{2n}, x_0, x_n, 1)$.

We shall give a 1-finite automaton $M_h = (S_h, s_e, s_q, s_g, s_f, \lambda_h)$, called the semicircuit solution whose synchronization time for $(C_{2n}, x_0, x_n, 1)$ is $3n$-1 time units. M_h is essentially similar to the circuit solution M_c, and S_h includes S_c.

The evolution of the solution M_h on $(C_{2n}, x_0, x_n, 1)$ is depicted in Fig. 4, in which the evolution of M_c on $(C_n, x_0, 1)$ is also shown for the reference. It is shown that the signals generated at (z, t) in $(C_{2n}, x_0, x_n, 1)$ is identical to those at $(z, t-n)$ in $(C_n, x_0, 1)$ for $0 \leq z \leq n-1$ and $2n + z \leq t$.

Moreover, the center cell x_n fires at $t = 3n-1$. Hence all cells on the semicircuit $(x_0, \ldots, x_n)$ of C_{2n} fire at time $3n$-1 simultaneously. Fig. 5 gives the semicircuit solution for $(C_{12}, x_0, x_6, 1)$.

Next, we consider a d-quasi-circuit structure $(C^1_{2n}, x_{00}, X_n, d)$ in which X_n is the set $\{x_{nj}\}$ of the center of circuits in C^1_{2n} and all x_{nj}'s are designated. A solution for synchronizing all cells in X_n and all cells x_{ij}'s, where $0 \le i \le n-1$, $0 \le j \le h_i$, is given by an d-finite automaton M_h^d which is obtained by slightly modifying M_h.

M_h^d is defined from M_h by the same way as the quasi circuit solution M_c^d is defined from the circuit solution M_c. That is, the state transition function of M_h^d is defined for such input letters that all components other than the external signal are identical signals, and for these input letters M_h^d behaves as M_h does.

By the similar arguments used for proving Theorem 3.2, it is is easily shown that M_h^d is a solution for the above problem and its synchronization time for $(C_{2n}^1, x_{00}, X_n, d)$ is $3n-1$ time units. M_h^d is called the quasi-semicircuit solution.

Now, we shall give a $3r+1$ solution M_{3r+1}^d for the class of d-graph structures (G_r, x_g, d) by using the concept of M_{4r}^d and M_h^d. The state set S_{3r+1} of M_{3r+1}^d is expressed as $(S_1 \times S_2 \times\times S_{30} \times S_{40}) \cup \{s_f\}$ where S_1 and S_2 is the same set as S_{4r} of M_{4r}^d, $S_{30} = S_{40}$ is the same set as S_h^d of M_h^d, and s_f is the firing state.

Given a d-graph structure (G_r, x_g, d), M_{3r+1}^d starts at $t = 0$ to reduce (G_r, x_g, d) to a d-quasi-cirsuit structure (C_{2r}^1, x_g, d) as M_{4r}^d does and also starts at $t = 1$ to simulate the quasi-semicircuit solution M_h^d. The synchronizing time of M_{3r+1}^d appears to be $3r$ time units, but it is not.

Let x be a terminal cell for which $dist_G(x_g, x) = r' < r$. Since x moves to $(H_0, *, *, *)$ at time r' and moves to $(I, *, *, *)$ at time $r' + 1$ for M_{4r}^d, x moves to $(I, *, P_{00}', P_{00}')$ at time $r' + 1$ for M_{3r+1}^d. Then the subcells of x move to the firing state F of M_h^d at time $3r' < 3r$, while all other subcells move to F at time $3r$. Thus it fails to synchronize all the first subcells of G_r.

Since x has at least one adjacent cell *e.g.* i th cell with $m_i = 1$ which does not move to F prior to time $3r$. (m_i is the i th component of $(m_1, \cdots, m_d) \in S_2$ of x.) We define the firing of M_{3r+1}^d as follows. Let x be any cell in (G_r, x_g, d). When the first subcell of x is F and the first subcells of all adjacent cells of x are also in F at time t, x moves to the firing state s_f of M_{3r+1}^d at time $t + 1$. It requires one more time unit.

Theorem 4.2 The d-finite automaton M_{3r+1}^d is a solution for Π^d and its synchronization time for $(G_r, x_g, d) \in \Pi^d$ is $3r+1$ time units.

5. A $3r$ solution for d-graph structures.

In this section, we give a improved solution M_{3r}^d for Π^d whose synchronization time for $(G_r, x_g, d) \in \Pi_s^d$ is $3r$ time units, where Π_s^d is a subclass of Π^d.

We call a cell x in G_r, for which $dist_{G_r}(x_g, x) = r$, a radial cell. A cell x in G, for which there exists no cell such that $dist_G(x_g, y) \geq dist_G(x_g, x)$, is called a solitary cell.

The reason why M_{3r+1}^d requires one more time unit for the synchronization is that the first subcells of non-radial terminal cells move to $F \in S_h$ before time $3r$. We shall consider to overcome this difficulty without loss of a time unit.

Let x be a terminal cell for which $dist_{G_r}(x_g, x) = r'$. In M^d_{3r}, the first and second subcells of x move respectively to A and $P'_{00} \in S_h$ at time $r'+1$. This is achieved by slightly modifying M^d_{3r+1}. In other words, the first subcell behaves as if x is non-radial and the second one does as if x is radial.

For a non-radial terminal cell y, the first subcell of y moves to F at time $3r$ and the second one moves to F at time $3r'$. For a radial cell x, the first subcell of x does not move to F prior to time $3r$ and the second one moves to F at time $3r$. For a non-terminal cell, the first subcell moves to F at $t = 3r$.

From the above consideration, if each cell x recognizes prior to time $3r'$ ($r' = dist_{G_r}(x_g, x)$) whether it is radial or not then all cells can fire once at time $3r$.

Usually, a terminal cell x requires r' time units ($r' = dist_{G_r}(x_g, x)$) to recognize that it is terminal and hence requires $3r'$ time units to recognize whether it is radial or not. But a solitary non-radial cell x requires $r'-1$ time units to recognize that it is non-radial or there exists at least one non-solilary terminal cell y such that $dist_{G_r}(x_g, y) = r'$. If all radial cells are solitary, then each of them recognizes at time $3r-1$ that it is radial and all other cells recognize prior to time $3r-1$ that all radial cells are solitary. Thus for (G_r, x_g, d) in which every radial cell is solitary, we can obtain a solution whose synchronization time is $3r$ time units.

Let Π^d_s be a subclass of Π^d consisting of d-graph structures in which all radial cells are solitary. We shall give a solution

M^d_{3r} whose synchronization time for (G_r, x_g, d) of Π^d_s and $\Pi^d - \Pi^d_s$ are respectively $3r$ and $3r+1$ time units.

The fundamental behaviors of M^d_{3r} is identical to that of M^d_{3r+1}. M^d_{3r} simulates the reduction from a d-connected graph structure to a d-quasi circuit structure and then simulates the behaviors of the quasi-semicircuit solution. Beside these behaviors, when the general cell recognizes that all radial cells are solitary, it sends signals about this knowledge to all other cells.

The state set of M^d_{3r} is $(S_{10} \times S_2 \times S_3 \times S_4) \cup \{s_f\}$. $S_{10} = S_1 \cup \{G_{20}, I_0, I_1, I_2, I_{20}, J_0, J_1, J_2, J_{20}\}$. S_1 and S_2 are given in M^d_{4r} and $S_3 = S_4 = s_h$ are given in M_h. s_f is the firing state of M^d_{3r}. The states in S_{10} play the following roles. The general cell moves to G_2 when it recognizes all radial cells to be solitary, or moves to G_{20} when it finds at least one non-solitary radial cell. If a terminal cell x recognizes itself to be solitary, then x moves to I_2 via I_0 and I_1, else x moves to I. When x in I_2 recognizes itself not to be radial or it finds at least one non-solitary terminal cell y such that $dist_{G_r}(x_g, y) = dist_{G_r}(x_g, x)$, x moves to I. When a cell in I recognizes that all radial cells are solitary, it moves to I_{20}. J, J_0, J_1 and J_2 play the same roles for non-terminal cells as I, I_0, I_1 and I_2 do for terminal cells. When a cell in J_2 recognizes that all radial cells are solitary, it moves to J_{20}. Thus, for any d-graph structure in Π^d_s, the general cell, radial cells, non-radial terminal cells, and the non-terminal cells move respectively to G_{20}, I_2, I_{20}, and J_{20} one time unit before their firing.

Signals in S_1 play the same roles for M^d_{3r} as it does for M^d_{4r}. Signals I_0, I_1, and I_2 are generated at solitary cells. Signals G_{20}, J_{20}, and I_{20} are generated when all radial cells are found to be solitary and are used to transmit this knowledge. Initially, the general cell is in the state (G_0, (0, ···, 0) Q, Q) and other cells are in (Q_0, (0, ···, 0), Q, Q).

In simulating the behaviors of the d-quasi-semicircuit solution M^d_h, M^d_{3r} makes each subcell to receive input signals only from its predecessor subcells and ignore input signals from other subcells. This can be achieved by the method similar to those used in M^d_{4r}.

If a cell x in (G_r, x_g, d) with $dist_G$ (x_g, x) = r' is solitary, x moves to I_0 at time r' and sends a J_0 series to the general cell x_g. Thus, if all radial cells are solitary, then x_g moves to G_{20} at time $2r$ and sends J_{20} signals to all cells in (G_r, x_g, d), else x_g moves to G_2 at time $2r$+1. As a result, if all radial cells are solitary, they are in I_2 and all other cells are in G_{20}, J_{20}, or I_{20} at time $3r$-1. (For r = 1, radial cells are in I_1, instead of I_2.) On the otherhand, if not all of radial cells are solitary, then solitary radial cells are in I_2 and all other cells are in G_2, J, or I at time $3r$-1. (For r = 1, solitary radial cells move to I_1.)

Note that if all radial cells are solitary, then they can recognize themselves to be radial at time $3r$-1. Hence we define the firing of M^d_{3r} as follows.

(1) If x is in G_{20}, J_{20}, or I_{20} at time t and the first

subcell of x moves to $F \in S_{30} = S_h$ at time $t+1$, then x moves to s_f at time $t+1$.

(2) If x is in I_2 or I_1, all x_i's with $m_i = 1$ are in J_{20} or G_{20} at time t, and the second subcell of x moves to F at time $t+1$, then x moves to s_f at time $t+1$.

(3) If x is in G_2 or J and the first subcell of x moves to F at time t, then x moves to s_f at time $t+1$.

(4) If x is in I and the first subcells of x_i's with $m_i = 1$ move to F at time t, then x moves to s_f at time $t+1$.

If all radial cells are solitary, then all cells in $(G_r, x_g, 1)$ fire at time $3r$ according to (1) and (2), else all cells fire at time $3r+1$ according to (3) and (4). (See Figs. 6 and 7.)

Theorem 4.1 M^d_{3r} is a solution for Π^d. The synchronization time of M^d_{3r} is 1 time unit for $(G, x_g, d) \in \Pi^d$ with $|G| = 1$, $3r$ time units for $(G_r, x_g, d) \in \Pi^d_s$ with $|G| \geq 2$, and $3r+1$ time units for $(G_r, x_g, d) \in \Pi^d - \Pi^d_s$ with $|G| \geq 2$.

Finally, we shall show that M^d_{3r} give the minimum synchronization time for some subclasses of Π^d_s. Let (G_r, x_g, d) be a member of Π^d_s which has two radial cells such that $dist_G(x_1, x_2) = 2r$. We denote the set of such d-connected graph structures by Π^d_{3r}. Theorem 1.1 shows that

$$t_{min}(G_r, x_g, d) \geq L(G_r, x_g, d) = 3r$$

for any (G_r, x_g, d) in Π^d_{3r}. Obviously Theorem 4.1 gives the synchronization time of $3r$ time units for any (G_r, x_g, d) in Π^d_{3r}. Thus, M^d_{3r} is a minimum time solution for Π^d_{3r}.

Let (G_r, x_g, d) be a member of $\Pi^d - \Pi^d_s$ which has three radial cells x_1, x_2, and x_3 such that x_1 and x_2 are adjacent each other and $dist_{G_r}(x_1, x_3) = dist_G(x_2, x_3) = 2r$.

We denote the set of these d-connected graph structures by Π^d_{3r+1}. Theorem 1.1 shows that

$$t_{min}(G_r, x_g, d) \geq L(G_r, x_g, d) + 1 = 3r + 1$$

for any (G_r, x_g, d) in Π^d_{3r+1}. Theorem 4.1 gives the synchronization time of $3r+1$ time units for any (G_r, x_g, d) in Π^d_{3r+1}. Thus, M^d_{3r} is a minimum time solution for Π^d_{3r+1}. There results give Theorem 4.2.

Theorem 4.2 Let Π^d_m be $\Pi^d_{3r} \cup \Pi^d_{3r+1} \cup \{(G, x_g, d) \mid |G| = 1\}$. M^d_{3r} gives the minimum synchronization time for any (G_r, x_g, d) in Π^d_m.

Conclusion

We have examined solutions of the firing-squad synchronization problem for some classes of d-graph structures and graph structures.

In the first part, we have given solutions for the classes of circuit structures, quasi-circuit structures, and some other digraph structures.

In the second part, we have given two solutions for the class of connected graph structures whose synchronization time for (G_r, x_g, d) are respectively $4r$ and $3r+1$ time units where G_r is a graph with the radius r.

In the final part, we have given a improved solution for connected graph structures whose synchronization time for (G_r, x_g, d) is $3r$ or $3r+1$ time units depending upon the property of radial cells. Moreover, we have shown that our solution give the minimum synchronization time for a subclass of connected graph structures.

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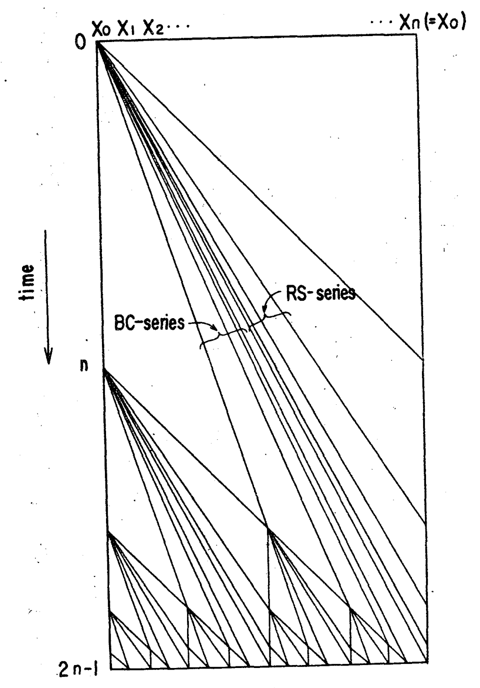


Fig. 1 The scheme of the solution M_c for $(C_n, x_0, 1)$.

time ↓

	0	1	2	3	4	5	6	7	8	9	10	11	12
0	P_{00}												
1	P_1	A_{00}											
2	P_0	B^2_2	A_{01}										
3	P_1	B_0	L^1	A_{00}									
4	P_0		C^2_1	T	A_{01}								
5	P_1	K	C_2	S^1_0		A_{00}							
6	P_0		C_0	B^2_2	S_1	T	A_{01}						
7	P_1	K		B_{13}	S^1_2	T		A_{00}					
8	P_0		K	B_2	B^2_2	R_2		T	A_{01}				
9	P_1	K		B_0	B_3	L^1	R_1	T		A_{00}			
10	P_0		K		C_1	B^2_2	R_2	T		T	A_{01}		
11	P_1	K		K	C_2	B_0	L^1	S_0		T		A_{00}	
12	P_0		K		C_0		C^2_1	T	S_1	T		T	A_{01}
13	P_{11}	K		K		B_1	C_2	S^1_0	S_2	T		T	
14	P_0	A_{10}	K		K	B_2	C_3	B^2_2	S_1	R_2		T	
15	P_1	B^2_1	A_{11}	K		B_0		C_{23}	S^1_2		R_1	T	
16	P_0	B_2	L^1	A_{10}	K		C_1	C_3	B^2_2	S_1	R_2	T	
17	P_1	B_0	L^2	T	A_{11}	K	C_2	K	C_{23}	S^1_2		S_0	
18	P_0		C_1	S^1_0		A_{10}	C_0		C_0	B^2_2	S_1	T	S_1
19	P_1	K	C_2	B^2_1	S_1	T	P_{20}			B_{13}	S^1_2	T	S_2
20	P_{00}		C_0	B_2	S^1_2	T	P_2	P_{00}		B_2	B^2_2	R_2	
21	P_1	A_{00}		B_{13}	L^2	R_2	P_2	P_1	A_{00}	B_3	B_3	L^1	R_1
22	P_0	B^2_2	A_{01}	B_2	B_2	L^1	P_{21}	P_0	B^2_2	A_{01}	B_2	B^2_2	R_2
23	P_{11}	B_0	L^1	P_{11}	B_3	L^2	P_2	P_{11}	B_0	L^1	P_{11}	B_3	L^1
24	P_0	P_{22}	C^2_1	P_0	P_{22}	B_2	P_{22}	P_0	P_{22}	C^2_1	P_0	P_{22}	B^2_2
25	F	F	F	F	F	F	F	F	F	F	F	F	F

$n = 13$, $S_g = P_{00}$, $S_f = F$

S_q is denoted by the blank.

Fig. 2 The solution M_c for $(C_{13}, x_0, 1)$.

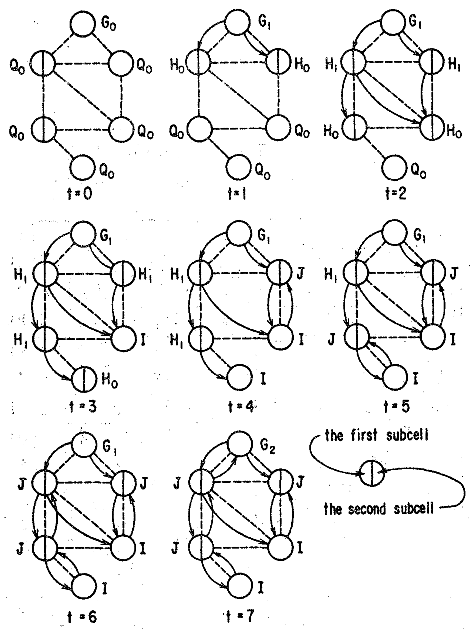


Fig. 3 The illustration of the solution M^d_{4r} for $(G_3, x_g, 3)$.

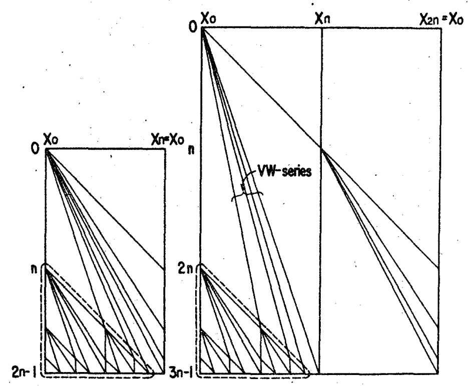


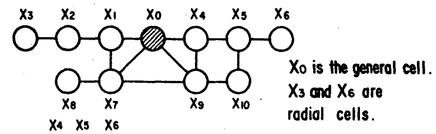
Fig. 4 The scheme of the solution M_h for $(C_{2n}, x_0, x_n, 1)$.

	0	1	2	3	4	5	6	7	8	9	10	11
0	P											
1	P'_0	A										
2	P'_1	V^1_1	A									
3	P'_2	V^2_2		A								
4	P'_3	V^3_3			A							
5	P'_0	V_4	Y^1			A						
6	P'_1	V_0	Y^2				P'_{00}					
7	P'_2		W^3_1				P''_{00}	A_{00}				
8	P'_3		W_2	V^1_1			P''_{00}	L^2	A_{01}			
9	P'_0		W_3	V^2_2			P''_{00}		L^1	A_{00}		
10	P'_1	K'	W_4	V^3_3			P''_{00}		L^2	T	A_{01}	
11	P'_2		W_0	V_4	Y^1		P''_{00}			S^1_0		A_{00}
12	P_{00}			V_{15}	Y^2		P''_{00}			L^2	S_1	T
13	P_1	A_{00}		V_2	V^3_3		P''_{00}				S^1_2	T
14	P_0	B^2_2	A_{01}	V_3	V_4	Y^1	P''_{00}				L^2	R_2
15	P_{11}	B_0	L^1	P_{11}	V_5	Y^2	P''_{00}					L^1
16	P_0	P_{22}	C^2_1	P_0	P_{22}	V^3_3	P''_{00}					L^2
17	F	F	F	F	F	F	F					

$n=6$, $S=P_g$, $S_f=F$

S_q is denoted by the blank.

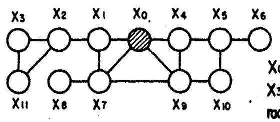
Fig. 5 The solution M_h for $(C_{2\cdot 0}, x_0, x_{10}, 1)$.



	X_0 / X_4	X_1 / X_5	X_2 / X_6	X_3		X_7	X_8		X_9	X_{10}
0	G_0									
1	G_1	H_0				H_0			H_0	
2	G_1	H_1	H_0			H_1	I_0		H_1	H_0
3	G_1	H_1	H_1	I_0		J_0	I_1		H_1	I
4	G_1	H_1	J_0	I_1		J_1	I_2		J	I
5	G_1	J_0	J_1	I_2		J	I_2		J	I
6	G_{20}	J_1	J_2	I_2		J	I		J	I
7	G_{20}	J_{20}	J_2	I_2		J_{20}	I		J_{20}	I
8	G_{20}	J_{20}	J_{20}	I_2		J_{20}	I_{20}		J_{20}	I_{20}
9	S_f	S_f	S_f	S_f		S_f	S_f		S_f	S_f

	X^2_0 / X^1_0	X^1_4 / X^1_1	X^1_5 / X^1_2	X^1_6 / X^1_3	X^2_6 / X^2_3	X^2_5 / X^2_2	X^2_4 / X^2_1		X^1_9 / X^1_7	X^1_{10} / X^1_8	X^2_{10} / X^2_8	X^2_9 / X^2_7
0												
1	P											
2	P'_0	A							A			
3	P'_1	V^1_1	A						V^1_1	A	P'_{00}	
4	P'_2	V^2_2		A	P'_{00}				V^2_2		P''_{00}	A_{00}
5	P'_3	V^3_3			P''_{00}	A_{00}			V^3_3		P''_{00}	L^2
6	P'_0	V_4	Y^1		P''_{00}	L^2	A_{01}		V_4	Y^1	F	
7	P_{11}	V_0	Y^2		P''_{00}		L^1		V_0	Y^2	F	
8	P_0	P_{22}	W^3_1		P''_{00}		L^2		P_{22}	W^3_1	F	
9	F	F	F	V^1_1	F				F	F	F	

Fig. 6 The solution M^d_{gr} for $(G_4, x_g, 3)$ whose all radial cells are solitary.



X_0 is the general cell. X_3, X_6, and X_{11} are radial cells, but X_3 and X_{11} are not solitary.

	X_0	X_1	X_2	X_3 (X_{11})	X_4	X_5	X_6	X_7	X_8	X_9	X_{10}
0	G_0										
1	G_1	H_0			H_0			H_0		H_0	
2	G_1	H_1	H_0		H_1	H_0		H_1	I_0	H_1	H_0
3	G_1	H_1	H_1	H_0	H_1	H_1	I_0	J_0	I_1	H_1	I
4	G_1	H_1	H_1	I	H_1	J_0	I_1	J_1	I_2	J	I
5	G_1	H_1	J	I	J_0	J_1	I_2	J	I_2	J	I
6	G_1	J	J	I	J_1	J_2	I_2	J	I	J	I
7	G_2	J	J	I	J	J_2	I_2	J	I	J	I
8	G_2	J	J	I	J	J	I_2	J	I	J	I
9	G_2	J	J	I	J	J	I	J	I	J	I
10	S_f	S_f	S_f	S_f	S_f	S_f	S_f	S_f	S_f	S_f	S_f

The diagrams for subcells are the same as in Fig.21

Fig. 7 The solution M^d_{3r} for $(G_4, x_g, 3)$ which has non-solitary radial cells.