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Homeomorphisms on a three dimensional handle

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Mcmillan proved that any two sets of generators for $\pi_1(H)$ are equivalent for an orientable handle H . We extend his result to the non-orientable case . These results are interesting in view of non-orientable Heegaard diagrams , in particular $P^2 \times S^1$. All manifolds considered are to be triangulated . All embeddings and homeomorphisms are to be piecewise linear .

Definition. Let H be a compact connected 3-manifold . We say that H is an orientable or non-orientable handle with genus n respectively when H is homeomorphic to $D^2 \times S^1 \# \cdots \# D^2 \times S^1$ or $M^2 \times I \# \cdots \# M^2 \times I$ where D^2 is a 2-disk , S^1 is a 1-sphere , M^2 is a Möbius band , I is a unit interval and # is a disk sum (boundary connected sum) . Note that $D^2 \times S^1 \# M^2 \times I$ is homeomorphic to $M^2 \times I \# M^2 \times I$.

Definition. Let H be a handle with genus n and $J_1, \cdots, J_n$ mutually disjoint simple closed curves on ∂H . We say that $[J_k]_{k=1}^{n}$ is a system of generators for $\pi_1(H)$ when S is connected and the inclusion homomorphism $\pi_1(S) \longrightarrow \pi_1(H)$ is onto where $S = \partial H - \bigcup_{k=1}^{n} \mathring{N}(J_k, \partial H)$ and $N(J_k, \partial H)$ is a regular neighborhood of J_k's in ∂H .

Definition. Let $[J_k]_{k=1}^{n}$, $[\tilde{J}_k]_{k=1}^{n}$ be two systems of generators for $\pi_1(H)$. We say that $[J_k]_{k=1}^{n}$ is equivalent to $[\tilde{J}_k]_{k=1}^{n}$ when there is a homeomorphism h of H such that $h(J_i) = \tilde{J}_i$ $(i=1,2, \quad ,n)$ and $h(H) = H$.

Definition. Let M be a compact 3-manifold . We say that M is irreducible when any two-sphere embedded in M bounds a 3-cell in M .

Now let M be a compact connected 3-manifold such that ∂M is non-empty . Then we have ;

Theorem 1. If M is irreducible and $\pi_1(M)$ is n-free , then M is an orientable or non-orientable handle with genus n . (Compare theorem 32.1 [5] and lemma in [3] and see lemma 1 in [8] .)

Next let H be an orientable handle with genus n and $[J_k]_{k=1}^{n}$, $[\tilde{J}_k]_{k=1}^{n}$ any two systems of generators for $\pi_1(H)$. Then the following lemma follows from Mcmillan's method .

Lemma 1. $[J_k]_{k=1}^{n}$ is equivalent to $[\tilde{J}_k]_{k=1}^{n}$.

Proof. See lemma 3 in [8] .

Hereafter suppose that H is a non-orientable handle with genus n and $J_1 , \ldots\ldots , J_m$ $(m \geq 1)$ are mutually disjoint simple closed curves in ∂H such that $S = \partial H - \bigcup_{k=1}^{n} \mathring{N}(J_k , \partial H)$ is connected and the inclusion homomorphism $\pi_1(S) \longrightarrow \pi_1(H)$ is onto .

Lemma 2. If at least one of $[J_k]_{k=1}^{m}$ is a non-orientable loop (let it be J_1) ,then there are two handles H_1 , H_2 such that $H = H_1 \# H_2$, the genus of H_1 is one , $H_1 \supset J_1$, and the genus of H_2 is $(n-1)$.

Proof. We prove the lemma by induction of the genus of H . At first it is trivial by lemma 2 in [8] when the genus of H is one . We may assume that the lemma is true when the genus of H is less than n and that the genus of H is n . Then we will verify that the lemma is true . Let d be the natural homeomorphism from H onto H^*, a disjoint copy of H . Then form the compact 3-manifold M by identifying points which correspond under $d/S = S^*$. Since the inclusion homomorphism $\pi_1(S) \longrightarrow \pi_1(H)$ is onto , the inclusion homomorphism $\pi_1(H) \longrightarrow \pi_1(M)$ is onto by van Kampen [2] . It is also one-to-one since the identifying map is the natural homeomorphism of H . Hence $\pi_1(M)$ is also n-free . Now at least one of ∂M is a Klein bottle K since J_1 is non-orientable .

Consider the inclusion homomorphism $\pi_1(K) \longrightarrow \pi_1(M)$. Since $\pi_1(M)$ is n-free but $\pi_1(K)$ is not , the kernel of the inclusion homomorphism is non-trivial . By Loop theorem [6] and Dehn's lemma [5] , there is a 2-disk D in M such that $D \cap \partial M = D \cap K =$

∂D and ∂D is not homotopic to zero in K . We may assume that ∂D is $\partial N(J_1, \partial H)$, where $J_1 \subset K$, or a meridian circle of K by the lemma 1 in Lickorish [1] . Then the first case does not happen , since $\pi_1(M)$ is free . By the general position argument , $D \cap S$ consist of only one arc and simple closed curves . If all the simple closed curves are homotopic to zero in ∂H , then they are also homotopic to zero in S because of S being connected . Thus there is a 2-disk $\tilde{D}$ such that $\partial\tilde{D} = \partial D$ and $\tilde{D} \cap S$ is only one arc . Then $\tilde{D} \cap H = E$ is a 2-disk and $E \cap \partial H = \partial E$, $E \cap \bigcup_{k=1}^{m} J_k = E \cap J_1$ and $E \cap J_1$ is only one point . Let $N(E \cup J_1, H)$ be a regular neighborhood of $E \cup J_1$ in H . Then $N(E \cup J_1, H)$ is a non-orientable handle with genus one such that $\partial N(E \cup J_1, H) \supset J_1$. We set $H_1 = H - \mathring{N}(E \cup J_1, H)$, then $H = H_1 \# N(E \cup J_1, H)$. It is easy to see that H_1 is a handle with genus (n-1) by theorem 1 . Next if $D \cap S$ contain at least a simple closed curves which is not homotopic to zero in ∂H , then there is a 2-disk E in H (or H^*) such that $E \cap \partial H = \partial E$, $E \cap \bigcup_{k=1}^{m} J_k = \phi$ and ∂E is not homotopic to zero in ∂H . Two cases happen that ∂E separates ∂H into two components and otherwise .

Case(1). Soppose that ∂E separates ∂H into two components . Then by corollary 1.1 in [8] E separates H into two components

H_1 , H_2 . By theorem 1 , H_1, H_2 are handles with positive genus . (Since ∂E is not homotopic to zero in ∂H .) Thus $H = H_1 \# H_2$ and $\partial H_1 \supset J_1$ or $\partial H_2 \supset J_1$. Let ∂H_1 contain J_1 and $S_j = \partial H_j - \bigcup_{\substack{k=1 \\ \alpha_j \ni k}} N(J_k, \partial H_j)$ where $[J_k]_{\alpha_1 \ni k} \cup [J_i]_{\alpha_2 \ni i} = [J_\ell]_{\ell=1}^{m}$. Then S_j (j=1,2) is connected and H_j (j=1,2) is a retract of H . Then the inclusion homomorphism $\pi_1(S_j) \longrightarrow \pi_1(H_j)$ (j=1,2) is onto . Since the genus of H_j (j=1,2) is less than n , by induction there is a non-orientable handle with genus one such that it's boundary contains J_1 .

Case(2). Suppose that $\partial H - \partial E$ is connected . Then by lemma 4 $S - \partial E$ is connected . Hence there is a simple closed curves w which intersects ∂E with only one point , and which has no intersections with $[J_k]_{k=1}^{m}$. Let $N(E \cup w, H)$ be a regular neighborhood of $E \cup w$ in H . Thus $H = H_1 \# N(E \cup w, H)$ where $H_1 = H - \mathring{N}(E \cup w, H)$. By theorem 1 , H_1 is also a handle such that $\partial H_1 \supset J_1$. Since H_1 is a retract of H , the inclusion homomorphism $\pi_1(S_1) \longrightarrow \pi_1(H_1)$ is onto where $S_1 = \partial H_1 - \bigcup_{k=1}^{m} N(J_k, \partial H)$. Since the genus of H_1 is less than n , by induction there is a handle with genus one such that it's boundary contains J_1 . (Note that case (2) does not happen if m = n .) Q.E.D.

Lemma 3. Let $[J_k]_{k=1}^{n}$ be a system of generators for $\pi_1(H)$. Then at least one of $[J_k]_{k=1}^{n}$ is non-orientable .

Proof. Since the inclusion homomorphism $\pi_1(S) \longrightarrow \pi_1(H)$ is is onto , S is non-orientable . Now we may assume that all of $[J_k]_{k=1}^{n}$ are orientable . Then S is embedded in a 2-sphere since S is connected , the Euler characteristics of ∂H is 2-2n and all of $[J_k]_{k=1}^{n}$ are orientable . It contradicts that S is non-orientable . Q.E.D.

It is easy to verify the following theorem 2 from lemma 2 and lemma 3 and lemma 1 .

Theorem 2. Let H be a non-orientable handle with genus n and $[J_k]_{k=1}^{n}$, $[\tilde{J}_k]_{k=1}^{n}$ two systems of generators for $\pi_1(H)$ both of which contain the same number of orientable simple closed curves . Then $[J_k]_{k=1}^{n}$ is equivalent to $[\tilde{J}_k]_{k=1}^{n}$.

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