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Two-point boundary-value problems with a discontinuous semilinear term

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1. Introduction

In this paper we consider the equation

$$(1.1)\quad \begin{cases} d^2v/dx^2 + g(v) = 0 & \text{on } I = (0\,,\ell_0) \\ dv/dx(0) = dv/dx(\ell_0) = 0\,, \end{cases}$$

where g is a function with a discontinuity point of the first kind. Our purpose is to present all the solutions of (1.1) under some appropriate conditions on g and v. At the same time we show that the cardinality of the set of all the solutions is $\aleph_0$ (countably infinite).

The equation (1.1) appears in the following situation. Consider the degenerate parabolic system of u(x,t) and v(x,t),

$$(1.2)\quad \begin{cases} \partial u/\partial t = \qquad f(u\,,v) & \text{in } I\times(0\,,+\infty) \\ \partial v/\partial t = \partial^2 v/\partial x^2 + g(u\,,v) & \text{in } I\times(0\,,+\infty) \\ \partial v/\partial x(0\,,t) = \partial v/\partial x(\ell_0\,,t) = 0 & t>0 \\ \text{initial conditions.} \end{cases}$$

In a biological point of view, it may be considered that u (resp. v) represents the density of a plant(resp. a herbiore) for example. Consider steady-state solutions of (1.2). Then the first equation of (1.2) reduces an algebraic equation. Hence, u can be expressed as a multi-valued function of v. If we fix v_* and assign each of two different parts of the multi-valued function on each side of v_*, u becomes a function of v with discontinuity at v_*. Substituting the function u of v into the second equation of (1.2) and rewriting the composed function with the same symbol g, we obtain the equation (1.1). The existence of such a steady-state solution is recognized by some numerical experiments. (See Mimura [1]. Mimura also proved that stable steady-state solutions of (1.2) have a unique v_* determined by f and g under an appropriate assumption.)

2. Presentation of results

The semilinear term g we deal with in this paper is restricted to the one satisfying the following condition.

Condition 1.

(i) g is a function defined on (v_1, v_2) with a discontinuity point of the first kind v_* and satisfies

$$-\infty < g(v_*-0) < 0 < g(v_*+0) < +\infty .$$

(ii) $g<0$ in (v_1, v_*), >0 in (v_*, v_2), and g is Lipschitz continuous in $(v_1, v_*-0]$ and $[v_*+0, v_2)$. (i.e., $\hat{g}$ is Lipschitz continuous in $[v_1+\varepsilon, v_*]$ for any $\varepsilon>0$, where $\hat{g}=g(v)$ for $v \in (v_1, v_*)$ and $=g(v_*-0)$ at $v=v_*$. Similarly $[v_*+0, v_2)$ is considered.)

(iii) g is monotone decreasing in (v_1, v_*) and (v_*, v_2).

We rewrite (1.1) by the weak form:

$$(2.1) \quad \begin{cases} \text{Find } v \in H^1(I) \text{ such that} \\ (\, dv/dx \, , \, d\phi/dx \,) = (\, g(v) \, , \, \phi \,) \qquad \text{for all } \phi \in H^1(I), \end{cases}$$

where (,) is the inner product in $L^2(I)$. Our aim is to find all the solutions of (2.1) satisfying

$$(2.2) \qquad v_1 < v(x) < v_2 \qquad (\, 0 \leqq x \leqq \ell_0 \,) .$$

Remark 1. (2.2) makes sense since $v \in H^1(I)$ implies $v \in C(I)$ by Sobolev's lemma. Furthermore, for the solution v of (2.1), we have $v \in C^1(I)$, since Condition 1 gives $g(v) \in L^2(I)$ which yields $v \in H^2(I)$.

Theorem 1. *In the case $g(v_*) \neq 0$, all the solutions of (2.1) and (2.2)* *are* v_n^i, $i=1,2$ *and* $n \geqq n_0$, *where* $v_n^i(x)$ *are functions defined in the subsequent section (see (3.9)) and* n_0 *is a positive integer depending on* g *and* ℓ_0.

In the case $g(v_*) = 0$, v_* *is added to the solutions.*

Remark 2. If $g(v_1+0) = g(v_2-0) = 0$, and if g is Lipschitz continuous in $[v_1+0\,,\,v_*-0]$ and $[v_*+0\,,\,v_2-0]$, then $n_0 = 1$ for any $\ell_0 > 0$.

Remark 2 as well as Theorem 1 is proved at the next section.

3. Proof of Theorem 1.

Let $v(x\,;c)$ be a solution of the initial value problem:

$$(3.1)\qquad \begin{cases} d^2v/dx^2 = -\tilde{g}(v) & (\,x > 0\,) \\ dv/dx(0) = 0 & \\ v(0) = c, & \end{cases}$$

where $c > v_1$ and

$$\tilde{g}(v) = \begin{cases} g(v) & (\,v_1 < v < v_*\,) \\ g(v_*-0) & (\,v_* \leqq v\,). \end{cases}$$

Such $v(x\,;c)$ exists uniquely since $\tilde{g}$ is Lipschitz continuous in $[c-\varepsilon\,,+\infty)$ for some $\varepsilon > 0$ where any solution of (3.1) lies since $\tilde{g} < 0$.

Lemma 1. *Let* $v(x;c)$ *be as above. Then, we have*

$$(3.2)\qquad v(x\,;c_1) < v(x\,;c_2) \qquad (\,x \geqq 0\,)$$

for $v_1 < c_1 < c_2$.

Proof. Let $x_0 > 0$ be the first intersecting point of $v(x;c_1)$ and $v(x;c_2)$. Obviously it holds that

$$dv/dx(x_0;c_1) \geqq dv/dx(x_0;c_2).$$

On the other hand, by (iii) of Condition 1, we have

$$d^2v/dx^2(x;c_1) = -\tilde{g}(v(x;c_1)) \leqq -\tilde{g}(v(x;c_2)) = d^2v/dx^2(x;c_2) \qquad (\,0 < x < x_0).$$

Integrating both sides from 0 to x_0, we have

$$dv/dx(x_0;c_1) \leqq dv/dx(x_0;c_2).$$

Hence we obtain

$$(d/dx)^i v(x_0;c_1) = (d/dx)^i v(x_0;c_2) \qquad (i=0,1).$$

By the uniqueness of the initial value problem, we have

$$v(x;c_1) = v(x;c_2) \qquad (x \geqq 0).$$

This is a contradiction. Hence we obtain (3.2). Q.E.D.

Define a mapping ϕ from (v_1 , v_*) into $(0 , +\infty)$ by

$$v(\phi(c);c) = v_*.$$

Since $d^2v/dx^2(x;c) \geqq -g(c) > 0$, $\phi(c)$ is well-defined. By Lemma 1, $\phi(c)$ is monotone decreasing. Set

$$\bar{\ell} = \lim_{c \downarrow v_1} \phi(c).$$

Lemma 2. *ϕ is homeomorphic from (v_1 , v_*) onto $(0 , \bar{\ell})$ and strictly decreasing.*

Proof. ϕ is strictly decreasing and therefore injective by Lemma 1. We show ϕ is continuous. Let $\{c_j\}$ be any sequence in (v_1 , v_*) converging to $c \in (v_1 , v_*)$. It is sufficient for us to show that there exists a subsequence $\{c_{j_k}\}$ such that $\phi(c_{j_k})$ converges to $\phi(c)$. Let $\bar{c}$ (resp. $\underline{c}$) be the supremum (resp. infimum) of $\{c_j\}$. Since $\phi(c_j) \in [\phi(\bar{c}) , \phi(\underline{c})]$, there exists a subsequence $\{c_{j_k}\}$ which converges to some $d \in [\phi(\bar{c}) , \phi(\underline{c})]$. Then it holds that

$$\begin{aligned} &|v(d;c) - v_*| \\ &\leqq |v(d;c) - v(\phi(c_{j_k});c)| + |v(\phi(c_{j_k});c) - v(\phi(c_{j_k});c_{j_k})| \\ &\leqq |v(d;c) - v(\phi(c_{j_k});c)| + \max_{0 \leqq x \leqq \phi(\underline{c})} |v(x;c) - v(x;c_{j_k})| \\ &\to 0 \qquad \text{as } j_k \to +\infty , \end{aligned}$$

since $v(x;c)$ is continuous and $\tilde{g}$ is Lipschitz continuous in $[\underline{c} , v_* - 0]$.

We show ϕ is surjective. It is sufficient for us to show that

$$\phi(c) \to 0 \qquad \text{as } c \to v_*.$$

Suppose that $\lim_{c \uparrow v_*} \phi(c) = d > 0$. Fix $x_0 \in (0 , d)$. Then it holds that

$$v(x_0;c) \to v(x_0;v_*) > v_* \qquad \text{as } c \to v_* ,$$

since $v(x;c)$ converges to $v(x;v_*)$ uniformly in $[0, d]$. On the other hand, by $v(x_0;c) < v(d;c) < v_*$, we have

$$\limsup_{c\uparrow v_*} v(x_0;c) \leqq v_* .$$

This is a contradiction. Hence, $d = 0$.

That ϕ^{-1} is continuous is easily proved. Q.E.D.

For any $\ell \in (0, \bar{\ell})$, $v(x;\phi^{-1}(\ell))$ is a unique solution of

$$(3.3)\quad \begin{cases} d^2v/dx^2 = -g(v) & (0 < x < \ell) \\ dv/dx(0) = 0 \\ v(\ell) = v_* \end{cases}$$

satisfying

$$(3.4)\quad v_1 < v(x) < v_* \qquad (0 \leqq x \leqq \ell).$$

Uniqueness is proved easily by making use of Lemma 1. Define a mapping α from $(0, \bar{\ell})$ into $(0, +\infty)$ by

$$\alpha(\ell) = dv/dx(\ell;\phi^{-1}(\ell)).$$

Lemma 3. α *is strictly increasing and homeomorphic from* $(0, \bar{\ell})$ *onto* $(0, \bar{\alpha})$ *and satisfies*

$$(3.5)\quad 0 < \alpha(\ell) \leqq -g(v_*-0)\ell \qquad (0 < \ell < \bar{\ell}),$$

where $\bar{\alpha} = \limsup_{\ell\uparrow\bar{\ell}} \alpha(\ell)$.

Proof. We first show that α is strictly increasing. Fix ℓ_i, $i=1,2$, such that

$$0 < \ell_1 < \ell_2 < \bar{\ell} \quad \text{and} \quad \ell_2 < 2\ell_1 .$$

Set for $s \in [0, \ell_2]$

$$w_1(s) = v(\ell_1-s;\phi^{-1}(\ell_1)) \quad \text{and} \quad w_2(s) = v(\ell_2-s;\phi^{-1}(\ell_2)),$$

where $v(x;\phi^{-1}(\ell_1))$ is assumed to be extended in $x < 0$ to the even function. Then w_i, $i=1,2$, satisfies

$$(3.6)\quad \begin{cases} d^2w_i/ds^2 = -g(w_i(s)) & (0 < s < \ell_2) \\ w_i(0) = v_* \end{cases}$$

$$\Big\lfloor \; dw_i/ds(0) = -\alpha(\ell_i) \; .$$

For the purpose of an indirect proof, assume that $\alpha(\ell_1) \geqq \alpha(\ell_2)$. If $\alpha(\ell_1) = \alpha(\ell_2)$, (3.6) implies that $w_1(s) = w_2(s)$ $(0 \leqq s \leqq \ell_2)$, which contradicts that

$$w_1(\ell_2) > \phi^{-1}(\ell_1) > \phi^{-1}(\ell_2) = w_2(\ell_2) \; .$$

Assume $\alpha(\ell_1) > \alpha(\ell_2)$. Let s_0 be the first intersecting point of w_1 and w_2. Obviously it holds that

$$dw_1/ds(s_0) \geqq dw_2/ds(s_0) \; .$$

On the other hand, we have

$$d^2w_2/ds^2(s) = -g(w_2(s)) \geqq -g(w_1(s)) = d^2w_1/ds^2(s) \quad (0 \leqq s \leqq s_0) \; ,$$

since $w_2 \geqq w_1$ on $[0 \, , s_0]$ and g is monotone decreasing. Hence we obtain

$$\begin{aligned} dw_2/ds(s_0) &= -\alpha(\ell_2) + \int_0^{s_0} d^2w_2/ds^2 \, ds \\ &\geqq -\alpha(\ell_2) + \int_0^{s_0} d^2w_1/ds^2 \, ds \\ &> dw_1/ds(s_0) \; , \end{aligned}$$

which is a contradiction. Therefore we get $\alpha(\ell_1) < \alpha(\ell_2)$, which also shows α is injective.

(3.5) is proved easily by making use of

$$\begin{aligned} \alpha(\ell) &= \int_0^{\ell} d^2v/dx^2(x;\phi^{-1}(\ell)) \, dx \\ &= -\int_0^{\ell} g(v(x;\phi^{-1}(\ell)) \, dx \; . \end{aligned}$$

We next show that α is continuous. We have

$$\begin{aligned} |\alpha(\ell_1) - \alpha(\ell_2)| &\leqq |dv/dx(\ell_1;\phi^{-1}(\ell_1)) - dv/dx(\ell_2;\phi^{-1}(\ell_1))| \\ &\quad + |dv/dx(\ell_2;\phi^{-1}(\ell_1)) - dv/dx(\ell_2;\phi^{-1}(\ell_2))| \\ &\leqq |dv/dx(\ell_1;\phi^{-1}(\ell_1)) - dv/dx(\ell_2;\phi^{-1}(\ell_1))| \\ &\quad + \max_{0 \leqq x \leqq \ell_1+\varepsilon} |dv/dx(x;\phi^{-1}(\ell_1)) - dv/dx(x;\phi^{-1}(\ell_2))| \; , \end{aligned}$$

$$\text{for } 0 < \ell_1 < \ell_2 < \ell_1+\varepsilon < \bar{\ell} \; .$$

Letting $\ell_2 \to \ell_1$, we obtain $\alpha(\ell_2) \to \alpha(\ell_1)$ since $\phi^{-1}(\ell_2) \to \phi^{-1}(\ell_1)$.

That α is surjective and that α^{-1} is continuous are easily proved.

Q.E.D.

Define $s_1 = \alpha^{-1}$. Then, s_1 is homeomorphic from $(0 , \bar{\alpha})$ onto $(0 , \bar{\ell})$ and strictly increasing and satisfies

$$-\alpha/g(v_*-0) \leqq s_1(\alpha) \qquad (0 < \alpha < \bar{\alpha}).$$

$v(x;\phi^{-1}(s_1(\alpha)))$ is a unique solution of

$$(3.7)\quad \begin{cases} d^2v/dx^2 = -g(v) & (0 < x < s_1(\alpha)) \\ dv/dx(0) = 0 & \\ v(s_1(\alpha)) = v_* & \\ dv/dx(s_1(\alpha)) = \alpha & \\ v_1 < v(x) < v_* & (0 \leqq x \leqq s_1(\alpha)). \end{cases}$$

Similarly we can define s_2 such that s_2 is homeomorphic from $(0 , \bar{\bar{\alpha}})$ onto $(0 , \bar{\bar{\ell}})$ and strictly increasing and satisfies

$$\alpha/g(v_*+0) \leqq s_2(\alpha) \qquad (0 < \alpha < \bar{\bar{\alpha}}).$$

For any $\alpha \in (0 , \bar{\bar{\alpha}})$ there exists a unique solution of

$$(3.8)\quad \begin{cases} d^2v/dx^2 = -g(v) & (0 < x < s_2(\alpha)) \\ dv/dx(0) = 0 & \\ v(s_2(\alpha)) = v_* & \\ dv/dx(s_2(\alpha)) = -\alpha & \\ v_* < v(x) < v_2 & (0 \leqq x \leqq s_2(\alpha)) . \end{cases}$$

Set $\hat{\alpha} = \min(\bar{\alpha} , \bar{\bar{\alpha}})$ and $\hat{\ell} = \lim_{\alpha\uparrow\hat{\alpha}} (s_1+s_2)(\alpha)$. For $\alpha \in (0 , \hat{\alpha})$ we denote the unique solution of (3.7) (resp. (3.8)) by $v(x;\alpha,1)$ (resp. $v(x;\alpha,2)$). Then, s_1+s_2 is homeomorphic from $(0 , \hat{\alpha})$ onto $(0 , \hat{\ell})$ and strictly increasing and satisfies

$$\{-1/g(v_*-0) + 1/g(v_*+0)\}\alpha \leqq (s_1+s_2)(\alpha).$$

Let n_0 be the smallest positive integer greater than $\ell_0/\hat{\ell}$. Define α_n and v_n^i, $i=1,2$ and $n \geqq n_0$, by

$$(s_1+s_2)(\alpha_n) = \ell_0/n \quad ,$$

and

$$(3.9) \qquad v_n^i(x) = \begin{cases} v(x;\ \alpha_n, i) & (0 \leqq x \leqq s_i(\alpha_n)) \\ v((s_1+s_2)(\alpha_n) - x;\ \alpha_n, i+1) & (s_i(\alpha_n) < x \leqq (s_1+s_2)(\alpha_n)) \\ v_n^i(2(s_1+s_2)(\alpha_n) - x) & ((s_1+s_2)(\alpha_n) < x \leqq 2(s_1+s_2)(\alpha_n)) \\ \text{periodic with period } 2(s_1+s_2)(\alpha_n) & (2(s_1+s_2)(\alpha_n) < x \leqq \ell_0), \end{cases}$$

where $v(x;\ \alpha_n, 3)$ is equal to $v(x;\ \alpha_n, 1)$. It is easy to observe that v_n^i, $i=1,2$ and $n \geqq n_0$, satisfy (2.1) and (2.2). To complete the proof of Theorem 1 we must show that there exist no other solutions of (2.1) and (2.2). Let $v \not\equiv v_*$ be a solution of (2.1) and (2.2). By Remark 1 and (2.1), we have

$$(3.10) \qquad v \in C^1[0\,,\,\ell_0] \qquad \text{and} \qquad \int_0^{\ell_0} g(v(x))\,dx = 0.$$

We first show that there exists $x_0 \in (0, \ell_0)$ such that

$$(3.11) \qquad v(x_0) = v_* \qquad \text{and} \qquad dv/dx(x_0) \neq 0 \;.$$

Take z_0 such that $v(z_0) \neq v_*$. Let x_0 be the nearest point to z_0 satisfying $v(x_0) = v_*$. Such x_0 is well-defined since $\{x\,;\, x \in (0, \ell_0),\ v(x) = v_*\}$ is not empty by (3.10). Without loss of generality, we may assume that

$$x_0 < z_0 \qquad \text{and} \qquad v(z_0) < v(x_0) \ (= v_*) \;.$$

From (2.1), we observe v satisfies the first equation of (1.1) in $(x_0\,,\,z_0)$. Integrating the equation from x_0 to y_0, where $y_0 \in (x_0\,,\,z_0)$ is a point satisfying $dv/dx(y_0) < 0$, we have

$$\begin{aligned} dv/dx(x_0) &= dv/dx(y_0) + \int_{x_0}^{y_0} g(v(s))\,ds \\ &\leqq dv/dx(y_0) \\ &< 0 \;. \end{aligned}$$

Hence x_0 satisfies (3.11). Set $\alpha = -dv/dx(x_0) > 0$. While v is lying in $(v_1\,,\,v_*)$, v satisfies the first equation of (1.1). Therefore, v can be extended until v reaches v_* or $x = \ell_0$. In the former case there exists

$x_1 = x_0 + 2s_1(\alpha) \in I$ and satisfies

$$v(x_1) = v_* \quad \text{and} \quad dv/dx(x_1) = \alpha .$$

Since $v \in C^1(I)$ and $\alpha > 0$, $v(x)$ transverses $v = v_*$. While v is lying in (v_*, v_2), v satisfies the first equation of (1.1). Hence v can be extended until v reaches v_* or $x = \ell_0$. In the former case there exists $x_2 = x_1 + 2s_2(\alpha) \in I$ and satisfies

$$v(x_2) = v_* \quad \text{and} \quad dv/dx(x_2) = -\alpha .$$

Repeating this process on both sides of x_0, and noting the boundary condition, we observe that α must be equal to some α_n and that $v = v_n^1$ or v_n^2. This completes the proof of Theorem 1.

Proof of Remark 2. Under the conditions of Remark 2, the equation (3.1) with $c = v_1$ and $\tilde{g}(v_1) = 0$ has a unique solution $v = v_1$. Since $\tilde{g}$ is Lipschitz continuous in $[v_1, v_*]$, we have $\bar{\ell} = +\infty$. Similarly $\bar{\bar{\ell}} = +\infty$ is obtained. Therefore, we get $n_0 = 1$. Q.E.D.

Reference

[1] Mimura, M., Striking patterns in a diffusion system related to the Gierer and Meinhardt model, in preparation.