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On the spectra of certain distance-regular graphs

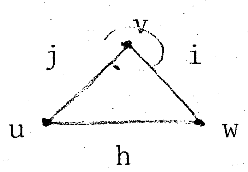
Eiichi Bannai (Gakushuin Univ.) and Tatsuro Ito (Tsukuba Univ.)

In this paper we discuss the unimodal property of distance-regular graphs.

We begin with the definition of distance-regular graphs and the unimodal property. A graph $\mathcal{G} = (V, E)$ is by definition a pair of sets V, E such that $E \subseteq V \times V$, $|V| < \infty$, $E = E'$ and $E \cap \Delta = \emptyset$, where $E' = \{(u, v) \mid (v, u) \in E\}$ and $\Delta = \{(v, v) \mid v \in V\}$. Elements of V and E are called vertices and edges respectively. For $u, v \in V$, $d(u, v)$ denotes the length of a shortest path joining u and v (∞ unless there is such a path), and d denotes $\max_{u,v \in V} d(u, v)$; $d(u, v)$ and d are called the distance between u and v and the diameter of $\mathcal{G}$ respectively. For $u, w \in V$ and integers $i, j, k \in \{0, 1, 2, \ldots, d\}$, let

$$P^i_{jh}(u, w) = \#\{v \in V \mid d(u, v) = j,\ d(v, w) = i\},$$

where $h = d(u, w)$.



Definition $\mathcal{G}$ is a distance-regular graph if each $P^i_{jh}(u,w)$ is independent of the choice of u, w with $h = d(u, w)$.

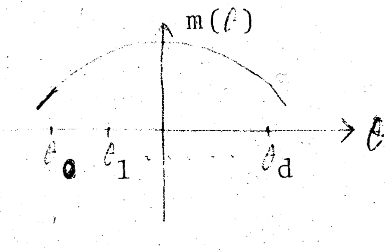
In the following we always assume that $\mathcal{G}$ is a distance-regular graph with diameter $d < \infty$ and $|V| = n$, and we simply write P^i_{jh} instead of $P^i_{jh}(u, w)$. P^1_{10} is called the valency and denoted by k .

Let $A = (a_{uv})$ be the $n \times n$ matrix defined by

$$a_{uv} = \begin{cases} 1 , & \text{if } (u, v) \in E \\ 0 , & \text{otherwise.} \end{cases}$$

Let $B = (b_{ij})$ be the $(d+1) \times (d+1)$ matrix defined by $b_{ij} = p^1_{ij}$. A and B are called the <u>incedence matrix</u> and the <u>intersection matrix</u> respectively. A is a symmetric matrix with size n , and B is a tri-diagonal matrix with size $d+1$ which has column sum k . Let us recall some well known properties of A and B . The algebra over $\mathbb{C}$ spanned by A is isomorphic to that spanned by B by the correspondence $A \longmapsto B$ (this is the regular representation of $\langle A \rangle$) . A has $d+1$ distinct eigenvalues and they are all real. Let $\theta_0, \theta_1, \ldots, \theta_d$ with $\theta_1 < \theta_2 < \cdots < \theta_d$ be the distinct eigenvalues of A and $m(\theta_i)$ $i = 0, 1, \ldots, d$ be the multiplicity of θ_i in A . Then $\theta_d = k$ and $m(\theta_d) = 1$.

<u>Definition</u> $\mathcal{G}$ has the <u>unimodal property</u> if there exists some i_0 such that $m(\theta_0) < m(\theta_1) < \cdots < m(\theta_{i_0})$ and $m(\theta_{i_0+1}) > m(\theta_{i_0+2}) > \cdots > m(\theta_d)$.



We are convinced that the problem below is the first step to the classification of distance-regular graphs.

<u>Problem</u> Classify distance-regular graphs with intersection matrix:

$$B = \begin{pmatrix} 0 & 1 & & & & \\ k & 0 & 1 & & & \\ & k-1 & 0 & \ddots & & \\ & & k-1 & \ddots & 1 & \\ & & & \ddots & 0 & c \\ & & & & k-1 & k-c \end{pmatrix} \quad \text{with } c \text{ integer.} \qquad \ldots\ldots\ (*)$$

So far the following results are known.

Theorem (Damerell-Bannai-Ito) If $c = 1$ and $d \geq 3$ and $k \geq 3$, then there exist no such graphs.
(If $c = 1$ and $d = 2$, then the classification has been completed except the case $k = 57$.)

Theorem (Georgiacodis [2]) If k = even and $d \geq 12$, then there exist no such graphs.

It seems to us that distance-regular graphs have the unimodal property in general (but not always) and the property greatly contributes to their classification. For example, our main theorem below helped Georgiacodis proving his theorem.

Theorem ([1]) Let $\mathcal{G}$ be a distance-regular graph whose intersection matrix B is of the form $(*)$. Then $\mathcal{G}$ has the unimodal property.

Corollary $[\mathbb{Q}(\theta_i) : \mathbb{Q}] \leq 2$ for $i = 0, 1, \ldots, d$.

Proof If θ_i and θ_j are algebraically conjugate, then $m(\theta_i) = m(\theta_j)$.

Three distinct eigenvalues cannot have the same multiplicity owing to the theorem and so cannot be algebraically conjugate.

Outline of the proof of the Theorem (For the details, see [1].) As is well known, the minimal polynomial of A is the same as that of B and has a factor $(x-k)$. Let $(x-k)F_d(x)$ be the minimal polynomial of B, and $(x-k)F_{d-1}(x)$ be that of B' , where

$$B' = \begin{pmatrix} 0 & 1 & & & & \\ k & 0 & 1 & & & \\ & k-1 & 0 & \ddots & & \\ & & k-1 & \ddots & 1 & \\ & & & \ddots & 0 & 1 \\ & & & & k-1 & k-1 \end{pmatrix} \quad \text{with size } d .$$

Then applying the general theory of tridiagonal matrices with column sum k , we have

$$m(\theta) = \frac{nk(k-1)^{d-1}}{(k-\theta)F_{d-1}(\theta)F'_d(\theta)} \qquad \ldots\ldots\ (**)$$

for root θ of $F_d(x)$, where $F'_d(x)$ is the derivative of $F_d(x)$. Transforming $(**)$ modulo $F_d(\theta)$, we get $m(\theta) = Q(\theta)/P(\theta)$ for some polynomials $P(x)$, $Q(x)$ of degree no more than three. By elaborate and somewhat tedious calculation we get the unimodal property.

In general let

$$B_i = \begin{pmatrix} 0 & a_1 & & & & \\ k & b_1 & a_2 & & & \\ & c_1 & b_2 & \ddots & & \\ & & c_2 & \ddots & a_{i-1} & \\ & & & \ddots & b_{i-1} & a_i \\ & & & & c_{i-1} & b'_i \end{pmatrix}$$

for $i = 0, 1, \ldots, d$, where $a_j+b_j+c_j = k$ for $j = 0, 1, 2, \ldots, d$ and $b'_i = b_i+c_i$ for $i = 0, 1, 2, \ldots, d$. Let $(x-k)F_i(x)$ be the minimal polynomial of B_i . Then there is a recurrence formula for the $F_i(x)$, and by the formula we can regard the series of polynomials $F_0(x)$, $F_1(x)$,..., $F_d(x)$ as orthogonal polynomials with respect to certain discrete weight. The identity (**) suggests a relationship with the Christoffel number $(\text{const.}/F_{d-1}(\theta)F'_d(\theta))$ of orthogonal polynomials. (cf. [3]) It is known that the unimodal property holds for the Christoffel numbers of classical orthogonal polynomials (see [3]). This is one of the reasons that led us to the following conjecture:

Conjecture the unimodal property holds for much far wider classes of distance-regular graphs.

References

[1] E. Bannai and T. Ito : On the spectra of certain distance regular graphs, to appear in J. of Combinatorial Theory (B).

[2] M. A. Georgiacodis : On the impossibility of certain distance-regular graphs, to appear.

[3] G. Szegö : Orthogonal polynomials, AMS Colloq. Publ. 1939 (4th edition 1975).