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On the failure of the Thompson fatorization in p-constrained groups

(Preliminary report)

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Let p be a prime. We say that a finite group G is p-constrained if $C_G(O_p(G)) \leq O_p(G)$. This is equivalent to $F^*(G) = O_p(G)$, where $F^*(G)$ is the generalized Fitting subgroup of G. For any finite group X, let $\mathcal{O}_p(X) = \{A \leq X \mid A$ is maximal elementary abelian p-group of X$\}$ and $\mathcal{A}_p(X) = \{A \leq X \mid A$ is maximal abelian p-subgroup of X$\}$. Let $J_1(X) = \langle A \mid A \in \mathcal{O}_p(X)\rangle$ and $J_2(X) = \langle A \mid A \in \mathcal{A}_p(X)\rangle$. These two subgroups are Thompson subgroups of X.

Let G be a p-constrained group and let S be a Sylow p-subgroup of G. Let $Z_1 = \Omega_1(Z(S))$ and $Z_2 = Z(S)$. G is said to satisfy the <u>Thompson factorization</u> if $G = C_G(Z_1)N_G(J_1(S))$ or $G = C_G(Z_2)(N_G(J_2(S)))$. It is interest to the theory of finite simple gorups to determine when the Thompson factorization fails.

M. Aschbacher in his work "On the failure of the Thompson factorization in 2-constrained groups" preprint gives a partial answer to the weaker problem $G \neq \langle C_G(Z_1), N_G(J_1)\rangle$ for p = 2. In treating the odd prime case, it seems that one has to generalized glauberman's result in "Weakly closed elements of Sylow subgroups" (Math. Z. 1968). In this report, we show some results in this direction concerning the situation $G \neq \langle C_G(Z_2), N_G(J_2(S)\rangle$.

Let p be a prime. In the rest of this report we write $\mathcal{A}(X)$ to mean $\mathcal{A}_p(X)$. Let r be a natural number. Let L be a normal p-subgroup of a finite group G. For any subgroup T of G, define $\mathcal{A}(r,L,T) = \{A \mid A \in \mathcal{A}(T)$ and $|L \cap A| \geq r\}$ and $J(r,L,T) = \langle A \mid A \in \mathcal{A}(r,L,T)\rangle$. Note that if we take r = 1, then $\mathcal{A}(r,L,T) = \mathcal{A}(T)$ and $J(r,L,T) = J(T)$.

<u>Lemma.</u> $J(r,L,T) \trianglelefteq N(T)$ <u>and if</u> $J(r,L,T) \leq H$, <u>a subgroup of</u> T, <u>then</u> $J(r,L,T) = J(r,L,H)$.

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Hypothesis II. Let p be a prime and r a natural number. Let L be a normal p-subgroup of G. Assume

(II.1) M is a subgroup of G ;

(II.2) S a Sylow p-subgroup of G ;

(II.3) $N(J(r,L,S)) \leq M$;

(II.4) Whenever $O_p(G) \leq H \leq G$, $H \nleq M$, $S \cap H$ is a Sylow p-subgroup of H, and $S \cap H$ contains an element of $\mathcal{A}(r,L,S)$ not contained in $O_p(G)$, then $N_H(J(r,L,S \cap H)) \nleq M$.

Note that $L \leq O_p(G) \leq S \leq M$ by (II.2) and (II.3) and $L \leq O_p(G) \leq O_p(H) \leq S \cap H$ by (II.4).

We say that G is a J(r,L)-minimal group if there exist natural number r, subgroups L,S,M of G satisfy Hypothesis II and $C_G(Z(O_p(G))) \leq M \leq G$. We say that G is a J(r,L)-minimal subgroup of X if G is a J(r,L)-minimal group and $O_p(X) \leq G$, $L \trianglelefteq X$ and G contains an element of $\mathcal{A}(r,L,S^*)$ not in $O_p(X)$ for some Sylow p-subgroup S^* of X.

Theorem. Let p be a prime, r a natural number. Let L be a normal p-subgroup of G and S a Sylow p-subgroup of G. Let $M(r,L,S) = \{K \mid K$ is a subgroup of G containig L, and $K \cap S$ is a Sylow p-subgroup of K, and K is a J(r,L)-minimal subgroup of G$\}$. Then

$$G = \langle C_G(Z(S)),\ N_G(J(r,L,S)) \text{ and subgroups in } M(r,L,S)\rangle.$$

Hypothesis III. Assume

(III. 1) Hypothesis II

(III. 2) $L \leq Z(O_p(G))$

(III. 3) $r = \max\{|A \cap L| \mid A \in \mathcal{A}(S) \text{ and } A \nleq O_p(G)\}$.

Lemma. Assume Hypothesis III. Let $A \in \mathcal{A}(r,L,S)$. Then $[L,A,A] = 1$.

Theorem. Assume Hypothesis III. Let $P = O_p(G)$, $C = C_G(L)$, $W = L/L \cap Z(G)$. Suppose $C \leq M < G$. Then C/P is a p'-group, W is an elementary abelian p-group, and G/C acts faithfully on W. Moreover :

(a) There exists $A \in \mathcal{A}(r,L,S)$ such that $A \nleq O_p(G)$.

(b) There exists a field $K \leq End(W)$ such that W is a 2-dimensional vector space over K and G induces SL(2,K) on W.

(c) <u>If</u> p = odd <u>or</u> $|K| = 2$, <u>then</u> $L = (L \cap Z(G)) \times [L,G]$.

(d) <u>If</u> A <u>satisfies</u> (a) <u>and</u> K <u>satisfies</u> (b), <u>then</u> $|K| = |AC/C|$ <u>and</u> S = PA.

Finally we would like to point out the following lemma which may have indepent interest.

<u>Lemma</u>. <u>Let</u> V <u>be a group and</u> $G \leq \mathrm{Aut}(V)$. <u>Assume</u> $G = \langle A,B \rangle$, $[V,AA] = 1 = [V,B,B]$, <u>and</u> $C_V(G) \geq [V,V]$. Let $\bar{V} = V/C_V(G)$ and $X = [\bar{V}, G]$. <u>Then</u> $X = [\bar{V}, A] \times [\bar{V}, B]$.

<u>Corolary</u>. <u>Let</u> V <u>be a finite dimensional vector space over a field F and let</u> G <u>be a group of linear transformations</u>. <u>Suppose</u> $G = \langle A, B \rangle$, $C_V(G) = 1$ <u>and</u> $V = [V, G]$. <u>If</u> $[V,A,A] = 1 = [V,B,B]$, <u>then</u> $V = [V, A] \times [V, B]$.

This corollary shows that in the special case $B = A^g$ for some $g \in G$, we have to study the group

$$\left\langle \begin{pmatrix} I_d & I_d \\ 0 & I_d \end{pmatrix}, \begin{pmatrix} I_d & 0 \\ X & I_d \end{pmatrix} \right\rangle,$$

where X is a nonsingular square d × d matrix. In the finite case, this is a generalized situation of Dickson's theorem. It is also interest to study the case when the characteristic of F is zero.