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Hecke rings over arbitrary fields

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Let (G,B,N,R,U) be a split (B,N)-pair of characteristic p and rank n, and K be an algebraically closed field of characteristic p. Let KG be the group algebra of G over K and $\bar{U}=\sum_{u\in U} u$.

At the conference the author introduced a construction of bases of Hecke rings over arbitrary fields, but in this note we concentrate on proving the next theorem. Further properties of general endomorphism rings of induced linear representations, i.e., Hecke rings over arbitrary fields are found in [2], [4] and [5].

Theorem 2. Let (G,B,N,R,U) and K be as above. Let $\mathbb{E}=\mathrm{End}_{KG}(KG\bar{U})$, then $\mathbb{E}$ is a Frobenius algebra.

We use the same notation and definitions as in [5].

Lemma 1. Let $A\mathbb{E}$ be a one-dimensional right ideal in $\mathbb{E}$ generated by A. Then $A(\bar{U})$ is a weight element of weight $(\gamma|H, \gamma(A_{(w_i)})_{1\leq i\leq n})$ where γ is a linear character of $\mathbb{E}$ into K afforded by $A\mathbb{E}$, i.e., $Ax=A\gamma(x)$ for all $x\in\mathbb{E}$ and $\gamma(h)=\gamma(A_h)$ for all $h\in H$.

Proof.

$hA(\bar{U})=A(h\bar{U})=A(\bar{U}h)=AA_h(\bar{U})=\gamma(h)A(\bar{U})$ for $h\in H$, and

$uA(\bar{U})=A(u\bar{U})=A(\bar{U})$ for $u\in U$.

$\bar{U}_i(w_i)A(\bar{U})=A(\bar{U}_i(w_i)\bar{U})=A(\bar{U}(w_i)\bar{U}_i)=AA_{(w_i)}(\bar{U})=\gamma(A_{(w_i)})A(\bar{U})$ from [5, Lemma(1.4)]. Q.E.D.

Proposition 1. Let $\{\pi_i \mid 1\leq i\leq s\}$ be as in [5, Theorem(2.11)]. Then (i) $\mathbb{E}=\pi_1\mathbb{E}\oplus\cdots\oplus\pi_s\mathbb{E}$ is a decomposition of the right regular module $\mathbb{E}_{\mathbb{E}}$ into non-zero indecomposable submodules $\{\pi_i\mathbb{E}\}$;

(ii) $\pi_i\mathbb{E} \cong \pi_j\mathbb{E}$ if and only if $i=j$, for all $1\leqslant i,j\leqslant s$;

(iii) all the right irreducible representations of $\mathbb{E}$ are one-dimensional.

Proof.

(i) is clear from [3, Corollary(54.10)], because $\{\pi_i\}$ are primitive idempotents of $\mathbb{E}$ such that $1=\pi_1+\dots+\pi_s$.

(ii) is also clear, because $\mathbb{E}_{\mathbb{E}}$ has exactly s equivalent classes of irreducible representations and right principle indecomposable modules from [3, Corollary(54.14)] and [5, Theorem(2.11)].

(iii) Since $\mathbb{E}$ has s linear representations (see [5, Theorem (2.11)]), $\mathbb{E}$ has also s one-dimensional right modules, which are all the right irreducible representations of $\mathbb{E}$. Q.E.D.

Proposition 2. Let $\mathbb{E}=\mathrm{End}_{KG}(KG\bar{U})$.

(i) Let $A\mathbb{E}$ be a one-dimensional right ideal in $\mathbb{E}$ generated by A, then $A(KG\bar{U})$ is an irreducible left module of weight $(\gamma|H, \gamma(A_{(w_i)})_{1\leqslant i\leqslant n})$ where γ is a linear character of $\mathbb{E}$ afforded by $A\mathbb{E}$.

(ii) Let $\mathbb{E}A$ be a one-dimensional left ideal in $\mathbb{E}$ generated by A, then $A(\bar{U})\in\bar{U}KG$ and $A(\bar{U})KG$ is an irreducible right KG-module of weight $(\varphi|H, \varphi(A_{(w_i)})_{1\leqslant i\leqslant n})$ where φ is a linear character of $\mathbb{E}$ afforded by $\mathbb{E}A$.

Proof.

(i) Since $KG\bar{U}=\sum_{\chi\in \mathrm{Hom}(B,K^*)} \oplus KG\varepsilon(\chi)$ where $K^*=K-\{0\}$ and $\varepsilon(\chi)=\sum_{b\in B}\chi(b^{-1})b$ (see the proof of [5, Proposition(2.8)]), $A(\bar{U})=m_{\chi_1}+\dots+m_{\chi_t}$ where $m_{\chi_i}\in KG\varepsilon(\chi_i)-\{0\}$ for all $1\leqslant i\leqslant t$. Since $A(\bar{U})$ is a weight element from Lemma 1, m_{χ_i}'s are also weight elements of the same weight as of $A(\bar{U})$. From [1, Corollary(6.11)], m_{χ_i}'s generate irreducible modules KGm_{χ_i}'s in $KG\varepsilon(\chi_i)$. Since the socle of $KG\bar{U}$ is multiplicity-free (see [5, Proposition(2.8)]), t=1 and $A(\bar{U})=m_{\chi_i}$. Hence $A(KG\bar{U})$ is an irreducible module.

(ii) Since $A\in End_{KG}(KG\bar{U})$, we have $A(\bar{U})\in\bar{U}KG$ from [5, Proposition (1.5)]. Since $A_{(w_i)}A(\bar{U})=A(\bar{U})(w_i)\bar{U}_i^-$ and $A_hA(\bar{U})=A(\bar{U})h$, $A(\bar{U})$ is a right weight element of weight $(\varphi|H, \varphi(A_{(w_i)})_{1\leq i\leq n})$. Hence we can prove the assertion by the similar argument as in (i).Q.E.D.

Theorem 1. Let $\mathbb{E}=End_{KG}(KG\bar{U})$ and $\{\pi_i|1\leq i\leq s\}$ be the primitive idempotents as in [5, Theorem(2.11)]. Then,

(i) for all $1\leq i\leq s$

socle of the right ideal $\pi_i\mathbb{E}=KA({}^{w_o}J, {}^{w_o}\chi)$

where w_o is a unique element of maximal length in W, and

socle of the left ideal $\mathbb{E}\pi_i=KA(J,\chi)$,

where $A(J,\chi)\pi_i=A(J,\chi)$;

(ii) let $\mathbb{E}_o$ be the socle of the right regular ideal $\mathbb{E}_{\mathbb{E}}$, then $\mathbb{E}_o$ is also the socle of the left regular ideal ${}_{\mathbb{E}}\mathbb{E}$ and

$$\mathbb{E}_o=\sum_{(J,\chi)\in P}\oplus KA(J,\chi) .$$

Proof.

(i) Since $A(J,\chi)\pi_i=A(J,\chi)$ if and only if $\pi_iA({}^{w_o}J, {}^{w_o}\chi)=A({}^{w_o}J, {}^{w_o}\chi)$ for all $1\leq i\leq n$ and $(J,\chi)\in P$, it is clear that $\pi_i\mathbb{E}\supset KA({}^{w_o}J, {}^{w_o}\chi)$ if $A(J,\chi)\pi_i=A(J,\chi)$.

Let $\mathbb{M}$ be an irreducible right module contained in $\pi_i\mathbb{E}$, then $\mathbb{M}$ is one-dimensional, i.e., $\mathbb{M}=KA$ for some $A\in\mathbb{E}-\{0\}$. From (i) of Proposition 2 $A(KG\bar{U})$ is an irreducible module. Since $\pi_iA=A$, $A(KG\bar{U})\subset Y_i$ where $Y_i=\pi_i(KG\bar{U})$. Hence $A(KG\bar{U})=A({}^{w_o}J, {}^{w_o}\chi)(KG\bar{U})$, because the socle of $Y_i=A({}^{w_o}J, {}^{w_o}\chi)(KG\bar{U})$. From [1, Theorem(4.3)] we have $A(\bar{U})\in KA({}^{w_o}J, {}^{w_o}\chi)(\bar{U})$ and $KA=KA({}^{w_o}J, {}^{w_o}\chi)=\mathbb{M}$.

Again let $A(J,\chi)\pi_i=A(J,\chi)$, then $\mathbb{E}\pi_i\supset KA(J,\chi)$. Let $\mathbb{M}'$ be an irreducible left ideal of $\mathbb{E}$ contained in $\mathbb{E}\pi_i$, then $M'=KA'$ for some $A'\in \mathbb{E}-\{0\}$. From (ii) of Proposition 2 $A'(\bar{U})KG$ is an irreducible right KG-module contained in $\bar{U}KG$. Since the right socle of $\bar{U}KG$ is multiplicity-free, there exists a unique pair $(J',\chi')\in P$ such that $A'(\bar{U})KG= A(J',\chi')(\bar{U})KG$ and $KA'=KA(J',\chi')$. Since $A(J',\chi')\pi_i$ $=A(J',\chi')$, $(J',\chi')=(J,\chi)$.

(ii) is clear from (i). Q.E.D.

Proof of Theorem 2.

Let M be an irreducible left $\mathbb{E}$-module; then the dual module $M'=\mathrm{Hom}_{\mathbb{E}}(M,{}_{\mathbb{E}}\mathbb{E})$ is one-dimensional, because the socle of ${}_{\mathbb{E}}\mathbb{E}$ is multiplicity-free. Hence M' is irreducible. Similarly the dual module $N'=\mathrm{Hom}_{\mathbb{E}}(N,\mathbb{E}_{\mathbb{E}})$ is also one-dimensional and irreducible where N' is an irreducible right $\mathbb{E}$-module. From [3, Theorem(58.6)] we can conclude that $\mathbb{E}$ is a quasi-Frobenius algebra.

Since $\mathbb{E}$ is quasi-Frobenius, $\mathbb{E}$ and $(\mathbb{E}_{\mathbb{E}})^*=\mathrm{Hom}_K(\mathbb{E},K)$ have the same distinct indecomposable components. Since $\mathbb{E}$ is being decomposed into distinct indecomposable components $\mathbb{E}\pi_1,\ldots,\mathbb{E}\pi_s$ and $\dim_K\mathbb{E}=\dim_K(\mathbb{E}_{\mathbb{E}})^*$, $\mathbb{E}$ and $(\mathbb{E}_{\mathbb{E}})^*$ are isomorphic. Hence $\mathbb{E}$ is a Frobenius algebra. Q.E.D.

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