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MINIMALITY OF PAIRWISE SUFFICIENT σ-FIELDS

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Abstract

Two partial orders are defined for pairwise sufficient σ-fields. With respect to one of these partial orders we prove the existence of a pairwise minimum sufficient σ-field for the coherent families of Hasegawa and Perlman (Ann. Statist., 1974); this is used to prove their main result on the existence of the minimal sufficient σ-field for coherent families. With respect to the second partial order we prove the existence of infinitely many minimal pairwise sufficient σ-fields for the discrete case with an uncountable sample space.

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1. Introduction

Burkholder (1961, Corollary 3, p. 1197) showed that a minimal sufficient σ-field if it exists is essentially contained in every sufficient σ-field and hence is the minimum sufficient σ-field, i.e., necessary and sufficient in the sense of Bahadur (1954). Pitcher (1957) showed that minimum sifficient σ-fields do not exist always. However for the so-called compact case Pitcher (1965) proved the existence of minimum sufficient σ-fields. Pointing out a gap in his argument Hasegawa and Perlman (1974) gave a new proof of this result in a slightly more general set up. Similar questions relating to pairwise sufficient σ-fields seem worth exploring, particularly in the light of the comments in Section 5 of Hasegawa and Perlman (1974).

Let $\underline{P}$ be a fixed family of probability measures on a fixed measurable space $(\Omega, \underline{A})$ and recall that a σ-field $\underline{A}_s \subset \underline{A}$ is pairwise sufficient if $\underline{A}_s$ is sufficient for $\{P_1, P_2\}$ on $\underline{A}$, for all P_1, $P_2 \in \underline{P}$. To define minimality we introduce two partial orders on the family of sub-σ-fields of $\underline{A}$. All σ-fields considered in this note are assumed to be sub-σ-fields of $\underline{A}$.

Let $\underline{N}_P$ be the family of all P-null sets $A \in \underline{A}$. Let $\underline{N}_{\underline{P}} = \bigcap_{P \in \underline{P}} \underline{N}_P$ and $\underline{N}_{P_1, P_2} = \bigcap_{i=1,2} \underline{N}_{P_i} = N_{P_1 + P_2}$. Let $\underline{A}_1$, $\underline{A}_2$ be two sub-σ-fields. We shall write $\underline{A}_1 \subset \underline{A}_2(\underline{P})$ or $\underline{A}_1 \subset \underline{A}_2(I)$ if given $A_1 \in \underline{A}_1$, there exists $A_2 \in \underline{A}_2$ such that the symmetric difference $A_1 \Delta A_2 \in \underline{N}_{\underline{P}}$, i.e., $\underline{A}_1 \subset \underline{A}_2(I)$ if $\underline{A}_1 \subset \underline{A}_2 \vee N_{\underline{P}}$ where $\underline{A}_2 \vee \underline{N}_{\underline{P}}$ is the smallest σ-field containing $\underline{A}_2$ and $\underline{N}_P$. Similarly $\underline{A}_1 \subset \underline{A}_2(\{P_1,P_2\})$, P_1, $P_2 \in \underline{P}$ or $\underline{A}_1 \subset \underline{A}_2(II)$ if for any pair P_1, P_2 and $A_1 \in \underline{A}_1$ there exists $A_2 \in \underline{A}_2$ such that $A_1 \Delta A_2 \in \underline{N}_{P_1,P_2}$, i.e., $\underline{A}_1 \subset \underline{A}_2(II)$ if

$\underline{A}_1 \subset \bigcap_{P_1,P_2 \in \underline{P}} (\underline{A}_2 \vee \underline{N}_{P_1,P_2})$. Of these two partial orders the first is one of several due to Bahadur (1954) and is the one most frequently used for sufficient σ-fields. The second seems more natural for pairwise sufficient σ-fields. However for pairwise sufficient σ-fields $\underline{A}_1$, $\underline{A}_2$ it is easy to show

$\underline{A}_1 \subset \underline{A}_2(II)$ iff $\tilde{\underline{A}}_1 \subset \tilde{\underline{A}}_2$ where $\tilde{\underline{A}}_i = \bigcap_{P \in \underline{P}} (\underline{A}_i \vee \underline{N}_P)$. Thus restricted to the class of pairwise sufficient σ-fields the second ordering is equivalent to one also introduced by Bahadur (1954).

If in the family of all pairwise sufficient σ-fields. $\underline{A}_s$ is minimum (minimal) under the partial order (I) we shall say $\underline{A}_s$ is a minimum (minimal) pairwise sufficient σ-field. If the partial order (II) is considered then the terms used are pariwise minimum and pairwise minimal σ-fields. Thus $\underline{A}_s$ is pairwise minimum sufficient if it is pairwise sufficient and for every pair P_1, P_2, it is contained in all pairwise sufficient σ-fields up to the $\{P_1,P_2\}$-null sets $\underline{N}_{P_1,P_2}$. The relative position of pairwise indicates whether it refers to sufficiency alone or to minimality as well.

Certain facts follow rather easily from these definitions. Burkholder's (1961) argument applies to the second ordering and shows if a pairwise minimal sufficient σ-field exists it is actually pairwise minimum sufficient; see in this connection Remark 1 of Section 5. Clearly if $\underline{A}_1 \subset \underline{A}_2$(I) then $\underline{A}_1 \subset \underline{A}_2$(II) and hence minimal pairwise sufficiency implies pairwise minimal sufficiency which, as we just saw, is equivalent to pairwise minimum sufficiency. The converse implication is not true in general.

We now state our main results. Under the assumption of coherence of Hasegawa and Perlman (1974) we show a pairwise minimum sufficient σ-field always exists. This is used to fix the gap in Pitcher's (1965) proof pointed out by Hasegawa and Perlman (1974). In the discrete case with uncountable sample space of Basu and Ghosh (1969) we give a simple characterisation of the pairwise minimum sufficient σ-field; we also show that there exist infinitely many minimal pairwise sufficient σ-fields and hence no minimum pairwise sufficient σ-field exists.

2. Preliminaries

For any sub-σ-field $\underline{A}_1$, let

$$\check{\underline{A}}_1 = \bigcap_{P_1, P_2 \text{ in } \underline{P}} (\underline{A}_1 \vee \underline{N}_{P_1, P_2})$$

and

$$\tilde{\underline{A}}_1 = \bigcap_{P \text{ in } \underline{P}} (\underline{A} \vee \underline{N}_P).$$

Then

$$\underline{A}_1 \subseteq (\underline{A}_1 \vee \underline{N}_{\underline{P}}) \subseteq \check{\underline{A}}_1 \subseteq \tilde{\underline{A}}_1$$

$$\tilde{\tilde{\underline{A}}}_1 = \tilde{\underline{A}}_1, \qquad \check{\check{\underline{A}}}_1 = \check{\underline{A}}_1.$$

Moreover by Lemma 3.3 of Hasegawa and Perlman (1974) applied to each pair $\{P_1, P_2\}$,

$$\underline{A}_1 \text{ pairwise sufficient implies } \check{\underline{A}}_1 = \tilde{\underline{A}}_1 \ldots \qquad (*)$$

Lemma 3.3 also yields

$$\underline{A}_1 \text{ sufficient implies } \underline{A} \vee \underline{N}_{\underline{P}} = \tilde{\underline{A}} \ldots \qquad (**)$$

<u>Definition 1</u>. <u>For any probability measure</u> Q <u>the support of</u> Q <u>(relative to</u> $\underline{P}$) <u>is an</u> $\underline{A}$<u>-measurable set</u> S_Q <u>such that</u>

(i) $Q(S_Q) = 1$

(ii) <u>if</u> $A \in \underline{A}$, $A \subset S_Q$ <u>and</u> $Q(A) = 0$ <u>then</u> $P(A) = 0$ <u>for all</u> $P \in \underline{P}$.

Such a set doesn't exist always (vide Example 2 of Pitcher (1965)) but as Theorem 2 will show S_Q exists in the coherent case.

If $S_Q^{(i)}$, $i = 1, 2$, are both supports of Q then $S_Q^{(1)} \Delta S_Q^{(2)}$ is $\{Q\} \cup \underline{P}$-null.

If $\underline{P}$ is dominated and λ is an equivalent σ-finite measure then, i.e., $\lambda(A) = 0$ iff $A \in \underline{N}_{\underline{P}}$ then S_Q may be taken as the set where $dQ/d(Q + \lambda) > 0$. If S_Q exists its indicator function will be denoted by I_Q. Usually for any set A the indicator function will be denoted by I_A.

For each pair P_1, $P_2 \in \underline{P}$, let ϕ_{P_1,P_2} be any fixed version of $dP_1/d(P_1 + P_2)$.

Let F be a family of $\underline{A}$-measurable functions. Then the smallest σ-field with respect to which all elements in F are measurable is denoted by $\underline{A}(F)$.

<u>Proposition 1</u>. <u>Let</u> $\underline{A}_{ps}$ <u>be a sub-σ-field of</u> $\underline{A}$. <u>The following three conditions are equivalent</u>

i) $\underline{A}_{ps}$ <u>is pairwise sufficient for</u> $\underline{P}$ <u>on</u> $\underline{A}$.

ii) <u>For each pair</u> P_1, $P_2 \in \underline{P}$ <u>there exists an</u> $\underline{A}_{ps}$<u>-measurable function</u> $f_{P_1P_2}$ <u>such that</u> $f_{P_1P_2} = \phi_{P_1P_2}$ <u>a.e.</u>, $\{P_1, P_2\}$.

iii) $\underline{A}_{ps}$ <u>is pairwise sufficient for</u> $\underline{P}^*$ <u>in</u> $\underline{A}$ <u>where</u> $\underline{P}^*$ <u>is the family of countable convex combinations</u> $\sum\lambda_i P_i$ <u>of elements</u> $P_i \in \underline{P}$

<u>Proof</u>. Equivalence of (i) and (ii) follows form the factorisation theorem for dominated families. Equivalence of (i) and (iii) follows from the well-known equivalence of sufficency and pairwise sufficiency for dominated families. For an elegant proof see Speed (1975).

<u>Theorem 1</u>. <u>Suppose for each</u> $P \in \underline{P}$ <u>the support</u> S_P <u>relative to</u> $\underline{P}$ <u>exists</u>. <u>Let</u> $\psi_{P_1P_2} = \phi_{P_1P_2} \; (I_{P_1} + I_{P_2} - I_{P_1}I_{P_2})$, $F = \{\psi_{P_1P_2}, P_1, P_2 \in \underline{P}\} \cup \{I_p, P \in \underline{P}\}$ <u>and</u> $\underline{A}_{mps} = \underline{A}(F)$.
<u>Then</u> $\underline{A}_{pms}$ <u>is pairwise minimum sufficient</u>.

Before proceeding to the proof let us note that under the hypotheses of the theorem, $I_{P_1} + I_{P_2} - I_{P_1}I_{P_2}$ is just the indicator function of the support of $(P_1 + P_2)$ and $\psi_{P_1P_2}$ is a version of $dP_1/d(P_1 + P_2)$.

<u>Proof</u>. By Proposition 1(ii) and the preceding remark concerning $\psi_{P_1P_2}$, $\underline{A}_{pms}$ is pairwise sufficient. Suppose $\underline{A}_{ps}$ is any pairwise sufficient σ-field. In view of (*) it is enough to show $\underline{A}_{pms} \subset \underline{A}_{ps} \vee \underline{N}_P$ for all $P \in \underline{P}$. This will be the case if for each pair P_1, $P_2 \in \underline{P}$, $\psi_{P_1P_2}$ and I_{P_1} are $\underline{A}_{ps} \vee \underline{N}_P$ measurable for all $P \in \underline{P}$.

By Proposition 1(ii) and (iii) there exists an $\underline{A}_{ps}$-measurable function f_i such that f_i is a version of $d(P_i)/d(P_i + P)$. Let $B_i \in \underline{A}_{ps}$ be the set where $f_i > 0$. Clearly

$$\int_{B_i(\bar{\ })S^c_{P_i}} f_i d(P_i + P) = P_i(B(\bar{\ })S^c_{P_i}) = 0.$$

Since $f_i > 0$ in B_i, this implies

(2.1) $$P(B_i(\bar{\ })S^c_{P_i}) = 0.$$

Also $P_i(B_i^c(\bar{\ })S_{P_i}) \leq P_i(B_i^c) = 0$. But by definition of S_{P_i} this implies

(2.2) $$P(B_i^c(\bar{\ })S_{P_i}) = 0.$$

By (2.1) and (2.2)

(2.3) $$I_{B_i} = I_{P_i} \quad \text{a.e. } (P).$$

Hence

(2.4) $$I_{B_1} + I_{B_2} - I_{B_1}I_{B_2} = I_{S_{P_1+P_2}} \quad \text{a.e. } (P).$$

By Proposition 1(ii) there exists an $\underline{A}_{ps}$-measurable function g such that on $S_{P_1+P_2}$

$$g = \phi_{P_1 P_2} \quad \text{a.e.} \quad (P_1 + P_2)$$

and hence on $S_{P_1+P_2}$

(2.5) $$g = \phi_{P_1 P_2} \quad \text{a.e.} \quad (P).$$

By (2.3), (2.4) and (2.5)

(2.6) $$g.(I_{B_1} + I_{B_2} - I_{B_1 B_2}) = \psi_{P_1 P_2} \quad \text{a.e.} \quad (P).$$

It follows from (2.3) and (2.6) that I_{P_i} and $\psi_{P_1 P_2}$ are $\underline{A}_s \vee \underline{N}_P$-measurable. This completes the proof.

3. Coherent Families

Throughout this section we suppose $(\Omega, \underline{A}, \underline{P})$ is coherent in the sense of Hasegawa and Perlman (1974). Compactness in the sense of Pitcher (1965) implies coherence. It is an interesting open question whether coherence implies compactness. Note that there are serious gaps in the proof of Lemma 1.2 and Theorem 1.1 of Pitcher (1965). See in this connection Morimoto (1973, Appendix). If correct Pitcher's results would imply the equivalence of compactness and coherence. Using the symbols of Pitcher

(1965) we note that by his Lemma 1.2 which is correct for $p = \infty$, coherence of a family of measures M on a measurable space (X,S) implies the compactness of $B_\infty(X,S,M)$ in the $\varepsilon_1(X,S,M)$-topology where the last two terms are defined as in Pitcher (1965). Incidentally the application of Pitcher's Lemma 1.2 made by Hasegawa and Perlman (1975) is valid since all their functions are in $B_\infty(X,S,M)$.

It may be worth mentioning here that compactness is equivalent to a property called weak domination. See in this connection Yamada (1976) and L. Rogge (1972).

Theorem 2. For each $P \in \underline{P}$, the support S_p exists. Hence a pairwise minimum sufficient σ-field exists.

Proof. Fix P, $Q \in \underline{P}$; Q and P may be identical. Let A_Q be the set where $dP/d(P + Q) > 0$ and I_Q the indicator function of A_Q. Then $\{I_Q, Q \in \underline{P}\}$ is countably coherent. For take a sequence Q_i and let $Q_0 = \sum 2^{-i} Q_i$, $A_0 = \{w; dP/d(Q_0 + P) > 0\}$ and I_0 the indicator function of A_0. Then $I_{Q_i} = I_0$ a.e. $[Q_i]$. Since $(\Omega, \underline{A}, \underline{P})$ is coherent there exists an $\underline{A}$-measurable function f such that $f = I_Q$ a.e. $[Q]$, $Q \in \underline{P}$. Let $S = \{w; f > 0\}$. Then

$$I_S = I_P \quad \text{a.e. } [P]$$

$$= I_Q \quad \text{a.e. } [Q],\ Q \neq P,\ Q \in \underline{P}.$$

This implies S is the support S_P of P in $\underline{P}$

The last part of Theorem 2 now follows from Theorem 1.

Let $\underline{A}_{pms}$ be the pairwise minimum sufficient σ-field.

Proposition 2. If $\underline{A}_s$ is a sufficient σ-field then $\underline{A}_s \vee \underline{N}_{\underline{P}} \supset \tilde{\underline{A}}_{pms}$.

Proof. $\underline{A}_{pms} \subseteq \underline{A}_s \vee \underline{N}_{P+Q}$ for all P, Q in $\underline{P}$ which implies

$$\underline{A}_{pms} \subseteq \check{\underline{A}}_s = \tilde{\underline{A}}_s$$

by (*). So $\tilde{\underline{A}}_{pms} \subseteq \tilde{\tilde{\underline{A}}}_s = \underline{A}_s = \underline{A}_s \vee \underline{N}_{\_}$ by (**).

Theorem 3. $\tilde{\underline{A}}_{pms}$ is minimum sufficient.

The proof of this is the same as that of Theorem 2.5 of Pitcher (1965) except that we use Proposition 2 to complete the gap pointed out by Hasegawa and Perlman (1974). Since the proof is short it is reproduced for the sake of completeness.

Proof. $\tilde{\underline{A}}_{pms}$ is sufficient by the argument given in the proof of Theorem 2.5 of Pitcher (1965). The proof is completed by appealing to Proposition 2.

4. Discrete Case

We consider the discrete set up of Basu and Ghosh (1969). See also in this connection Morimoto (1972, 73), Kusama and Yamada (1972) and Brown (1975).

Suppose then $\underline{A}$ is the class of all subsets of Ω and $P \in \underline{P}$ is a discrete measure. We denote the probability function corresponding to P by p so that $P(A) = \sum_{w \in A} p(w)$. To avoid trifialities assume Ω is uncountable. We also assume for each $w \in \Omega$ there exists $P \in \underline{P}$ such that $P(\{w\}) > 0$.

As shown by Basu and Ghosh (1969) the minimal sufficient partition $\underline{D} = \{D\}$ is a family of disjoint sets of Ω whose union is Ω and which satisfies the collowing condition:

For any $D \in \underline{D}$, w_1, $w_2 \in D$ iff $p(w_1) > 0$ implies $p(w_2) > 0$ and $p(w_2)/p(w_1)$ is independent of p for all $P \in \underline{P}$ for which $p(w_1)p(w_2) > 0$. Note that each D is countable.

We say a σ-field $\underline{A}_1$ separates elements of $\underline{D}$ if given $D_1 \neq D_2 \in \underline{D}$ there exists $B \in \underline{A}_1$ such that $B \supset D_1$ and $B^c \supset D_2$.

Theorem 4. $\underline{A}_1$ is pairwise sufficient iff it separates elements of $\underline{D}$.

Proof. Suppose $\underline{A}_1$ is pairwise sufficient. Because of Proposition 1(iii) we assume without loss of generality $\underline{P}$ is closed under countable convex combinations. Hence given $D_1 \neq D_2$ we can find $P_0 \in \underline{P}$ such that

$$p_0(w) > 0 \text{ if } w \in D_1 \cup D_2. \tag{4.1}$$

Since $D_1 \neq D_2$ we can find $P \in \underline{P}$ and $K_1 \neq K_2$ such that

(4.2) $$p(w)/p_0(w) = K_i \quad \text{if} \quad w \in D_i, \quad i = 1, 2.$$

Let $A = \{w; p(w)/p_0(w) = K_1\}$. By Proposition 1(ii) $B \in \underline{A}_1$ such that $P_0(B \,\Delta\, A) = 0$. Hence by (4.2), $B \supset D_1$ and $B^c \supset D_2$, i.e., $\underline{A}_1$ separates elements in $\underline{D}$.

Conversely suppose $\underline{A}_1$ separates elements in $\underline{D}$. Choose P_1, $P_2 \in \underline{P}$. Then for any fixed K, $E = \{w; p_1(w)/(p_1(w) + p_2(w)) = K, p_1(w) + p_2(w) > 0\}$ is a countable union of elements in $\underline{D}$. Moreover $F = E^c \cap S_{P_1+P_2}$ is a countable union of elements in $\underline{D}$. Clearly $\underline{A}_1$ separates E and F and so there exists $B \in \underline{A}_1$ such that $B \supset E$ and $B^c \supset F$. So $\sum P_i(B \,\Delta\, E) = 0$. An appeal to Proposition 1 completes the proof.

Let $\underline{A}_0$ be the σ-field containing only countable union of elements in $\underline{D}$ and their complements.

Proposition 3. *For fixed $D \in \underline{D}$ let $\underline{A}(D)$ be the sub-σ-field of $\underline{A}_0$ containing all countable unions of elements of $\underline{D}$ which do not contain D and their complements. Let $\underline{A}_1$ be a proper sub-σ-field of $\underline{A}_0$. Then $\underline{A}_1$ separates elements of $\underline{D}$ iff there exists $D_0 \in \underline{D}$ such that $\underline{A}_1 = \underline{A}(D_0)$.*

Proof. Suppose $\underline{A}_1$ has the separating property. Since $\underline{A}_1$ separates elements of $\underline{D}$, if $\underline{A}_1$ contains a countable union $\cup D_i$, then each $D_i \in \underline{A}_1$. Since $\underline{A}_1 \subsetneq \underline{A}_0$, there exists a $D_0 \in \underline{D}$ such

$D_0 \not\in \underline{A}_1$. By our previous argument, $\underline{A}_1$ cannot contain any countable union $H = \cup D_i$ such that $D_0 \not\subset H$. By the separating property there exists $B \in \underline{A}_1$ such that $B \supset D_0$ and $B^c \supset H$. Obviously B must be uncountable and so B^c is countable implying $H \in \underline{A}_1$. So $\underline{A}_1$ contains all countable unions H which do not contain D_0. Thus $\underline{A}_1 = \underline{A}(D_0)$. The converse is obvious.

We can now state the main result in this section.

<u>Theorem 5.</u> <u>$\underline{A}_0$ is pairwise minimum sufficient. For each $D \in \underline{D}$, $\underline{A}(D)$ is minimal pairwise sufficient and hence no minimum pairwise sufficient σ-field exists.</u>

<u>Proof</u>. Note that $S_P = \{w; p(w) > 0\}$ is a countable union of elements in $\underline{D}$. Also if $\psi_{P_1P_2}$ is defined as in Section 2, then $\{w; \psi_{P_1P_2} < K\}$ is a countable union of elements in $\underline{D}$ if $K < 0$ and is the complement of such a set if $K > 0$. So $\underline{A}_0$ is identical with the σ-field $\underline{A}_{pms}$ defined in Theorem 1. This proves the first statement. The second statement follows immediately from Theorem 4 and Proposition 3, if we recall $\underline{N}_P$ is just the empty set here.

<u>Remark</u>. This result as well as Theorem 4 is valid if $\underline{A}$ is <u>any</u> σ-field separating points of Ω. Only slight changes in the proof are needed to show this.

5. Miscellaneous Remarks

Remark 1. Theorem 4 of Burkholder (1961) is trivially true if we replace $\underline{P}$ by a fixed pair $\{P_1, P_2\}$; this follows from Proposition 1(ii). Burkholder's (1961) Corollary 3 then follows with $\underline{P}$ replaced by $\{P_1, P_2\}$. This implies any pairwise minimal sufficient σ-field is pairwise minimum sufficient.

Remark 2. R. V. Ramamurthy has construced an example where no minimal pairwise sufficient σ-field exists.

Remark 3. In the set up of Section 4, B. V. Rao and R. V. Ramamurthy have shown there are many minimal pairwise sufficient σ-fields not covered by Theorem 5. But a complete description of all minimal pairwise sufficient σ-fields seems difficult.

Remark 4. It is perhaps worth pointing out that in the first example of Pitcher (1957) no minimum pairwise sufficient σ-field exists, but existence or non-existence of pairwise sufficient σ-fields of other types seems difficult to settle. In view of the remark following Theorem 5, infinitely many minimal pairwise sufficient σ-fields exist in his second example.

Remark 5. It would be interesting to know if pairwise minimum sufficient σ-fields exist in all problems.

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