SMALLESTNESS AND MINIMALITY OF PAIRWISE SUFFICIENT SUBFIELDS

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This is a continuation of the preceding article[1] by J.K. Ghosh. The same definitions and notations as in that article will be used here, except that the basic space $\Omega$ is now replaced by $X = \{ x \}$.

1. The smallest subfield with pairwise sufficiency and containment of carriers.

We begin with a simple example which, however, retains all the essential features of the discrete case in general.

[Example 1] Let $X$ be an uncountable space, $\mathcal{A}$ the sigma-field of all the subsets of $X$ and $\mathcal{P}$ be the family of all one-point probability measures on $X$. Define $\mathcal{C}$ to be the family of all the countable and cocountable sets and for each $x$ in $X$ define $\mathcal{C}(x)$ to be the family of all the countable sets which do not contain $x$ and all the cocountable sets which contain $x$. Clearly both of these families are sub-sigma-fields(simply,"subfields") of $\mathcal{A}$. It follows from Theorem 5 of [1] that $\mathcal{C}(x)$ is minimal pairwise sufficient(MPS, in short) and that $\mathcal{C}$ is pairwise minimum sufficient(PSS, as I would rather call it pairwise smallest sufficient).

I would like to point out that $\mathcal{C}(x)$ is also PSS. In fact, all the subfields which shares the same partition as a PSS subfield are also PSS, because in the discrete case, if two subfields have a same partition they are equivalent in terms
of the partial order (II) defined in [1]. Hence there are great many PSS in this case: All the separating subfields are PSS. Here, of course, a subfield $B$ is called separating if for any two points in $X$ there is a set in $B$ which contains one and only one of them. (Here the example ends).

To single out one "smallest" subfield, one other concept seems to me more convenient: The smallest subfield with pairwise sufficiency and containment of supports (SPSC). This is defined as the smallest (minimum, in terms of [1]) one wrt. the partial order (I), among all the pairwise sufficient subfields which contain supports of all $P$ in $P$ relative to $P$ itself, according to Definition 1 of [1]. In the foregoing example, $C$ is SPSC, as well as the particular PSS written $A_{pms}$ constructed in Theorem 1 of [1]. This coincidence is not an accident, as the following theorem shows.

[Theorem 1] Suppose for each $P$ in $P$ the supports relative to $P$ exists. Then $A_{pms}$ is SPSC.

[Outline of the proof] Assume that $B$ is pairwise sufficient. Then the functions $\psi_{P_1 P_2}$, defined in Theorem 1 of [1] must be $B$-measurable. If we assume that $B$ contains the supports of all $P$ in $P$, then the functions $I_P$ which appear in the same Theorem are $B$-measurable. Hence $B$ includes $A_{pms}$. (End)

Thus the existence of SPSC and its being PSS is proved under the same generality as the existence of the latter has thus far been proved. A possible criticism of this concept might...
be that while pairwise sufficiency is a "pairwise concept", that is, one preserved by the equivalence in terms of (II), the concept of support is not, and the definition of SPSC has a sort of inconsistency in its combining these two concepts belonging to two different categories. On the other hand, SPSC emerges quite naturally from the following theorem which holds under a slightly more general conditions than what is called weak domination.

[Generalized Neyman Factorization Theorem] (Yamada and Morimoto) A subfield B is pairwise sufficient and contains the supports of all P in P if and only if every P in P has a B-measurable density wrt. a pivotal measure.

Under the same generality, the existence of SPSC immediately follows: The subfield generated by all the versions of the densities of all P in P wrt. a pivotal measure is SPSC.

I would not state explicitly the conditions for the theorem or the definition of a pivotal measure here, because Neyman factorization is not the main subject here, and the existence of SPSC has been proved in [1] under a more general condition, that is, the existence of supports.

2. Characterization of minimal pairwise sufficient subfields in the discrete case.

I state in this section recent results by Namba[2]. I again take up Example 1, although the results are easily rephrased for the discrete case in general. Theorem 4 of [1] is now specialized to: A subfield is pairwise sufficient if and only if it is
separating. Thus our problem is to decide whether a given separating subfield is a minimal one of that kind or not. Suppose that \( \mathcal{B} \) is separating and let \( \mathcal{F} = \{ F_i; i \in I \} \) be a family of sets which generates \( \mathcal{B} \). Define \( 2^I \) to be the space of all functions on \( I \) to \( \{0, 1\} \). Here \( I \) is the set of indices attached to the sets in \( \mathcal{F} \). Points in this space are written \( y = (y(i); i \in I) \), \( z = (z(i); i \in I) \) etc. We define a mapping \( f \) on \( X \) onto a subset \( Y \) of \( 2^I \) as follows: A point \( x \) in \( X \) is mapped to a point \( y = f(x) \) which satisfies \( y(i) = 1 \) if \( x \) belongs to \( F_i \) and \( y(i) = 0 \) otherwise. By \( f \), \( \mathcal{F} \) is carried to the family of all such sets that are written as \( \{ y; y(i) = 1 \} \) for some \( i \) in \( I \).

And \( \mathcal{B} \) is carried by \( f \) to the sigma-field generated by it. We conveniently denote them by \( \mathcal{F} \) and \( \mathcal{B} \) again. A neighbourhood of \( y \) in \( Y \) is defined to be a set \( N(y; K) \), where \( K \) is any countable subset of \( I \), which is the totality of those points \( z \) in \( Y \) which satisfies \( z(i) = y(i) \) for all \( i \) in \( K \). The neighbourhoods, when \( y \) ranges over \( Y \) and \( K \) assumes to be all countable subsets of \( I \), give rise to a topology on \( Y \). \( Y \) is called \( \omega_1 \)-compact wrt. this topology if the following condition is satisfied: Assume that to each \( y \) in \( Y \) there corresponds a neighbourhood \( N(y; K(y)) \). Then one can choose a countable number of points \( y_0, y_1, \ldots, y_k, \ldots \) such that \( \bigcup_{k=0}^{\infty} N(y; K(y_k)) = Y \).

We now state a theorem of Namba[2] which gives a complete characterization of minimality of a separating subfield.

[Theorem 2] \( \mathcal{B} \) is minimal separating if and only if \( Y \) is \( \omega_1 \)-compact.
Let us see how this theorem works with the subfields given in Example 1.

[Example 2] Under the framework of Example 1, take the following generators $F$ and $F(x)$ of $C$ and $C(x)$, respectively:

$F = \{ \text{all the singletons in } X \}$.

$F(x) = \{ \text{all the singletons except } x \}$.

Corresponding sets of indices $I$ for these generators are $X$ and $X - \{x\}$, respectively. By the correspondence $f$, the space $X$ is mapped to one of the following two spaces, depending on cases:

$Y = \{ y \text{ such that } y(i) = 1 \text{ for one single } i \text{ in } I = X \}$.

$Y(x) = \{ y \text{ such that } y(i) = 1 \text{ for on single } i \text{ in } I = X - \{x\}, 0 \}$.

Here, $0$ denotes the point of $2^I$ such that $0(i) = 0$ for all $i$ in $I$. Notice that $f(x) = 0$.

The $\omega_1$-compactness of $Y(x)$ is proved as follows: Suppose that $N(y; K(y))$ corresponds to $y$, for each $y$ in $Y$. If a point $y$ does not belong to $N(0, K(0))$, the neighbourhood corresponding to $0$, then there exists $i$ in $K(0)$ such that $y(i) = 1$. As $K(0)$ is countable and as each $y$ can assume the value 1 for at most one $i$, there are a countable number of points which do not belong to $N(0, K(0))$. Take them as $y_1, y_2, \ldots, y_k, \ldots$ and 0 as $y_0$. Then it is clear that the neighbourhoods corresponding these points collectively cover $Y$.

This proof does not work for $Y$, because it does not contain 0. On the other hand it is easy to disprove $\omega_1$-compactness.
There exists an example of a minimal separating subfield which does not contain any singletons:

[Example 3] Let $I$ be an uncountable set of indices $i$ and for each non-negative integer $n$ let $Y_n$ be the set of all functions $I$ to $\{0,1\}$ which assume the value 1 for at most $n$ indices in $I$. Put $Y = \bigcup_{n=1}^{\infty} Y_n$, the set of all functions assuming the value 1 for a finite number of indices $i$ in $I$. Let $F$ be the family of all sets of the form: $\{y : y(i) = 1\}$ for some $i$ in $I$. Define $B$ as the subfield generated by $F$, which is equal to the totality of the sets $B$ for which there exists a countable subset $K(B)$ of $I$ such that $y \in B$ and $Y(i)=z(i)$ for all $i$ in $K(B)$ imply $y \in B$. Then it is clear that $B$ is separating and $B$ does not contain any singletons. To prove that it is minimal, we are sufficed to prove the $\omega_1$-compactness of $Y$.

The proof is similar in nature to that given in the previous example, except only that we need induction over $n$.

References
