数理解析研究所講究録
第 330 巻 1978 年 24-27

On complete Lašnev spaces

by

Takemi Mizokami (Sasebo Technical College)

Lašnev spaces are closed images of metric spaces, which are characterized by Stricklen Jr. in [3] as follows:

THEOREM 1. (Stricklen Jr. [3]) A space T is Lašnev iff T can be embedded in a regular (i.e. T_3) Fréchet space S with a dense metrizable subspace (A, d) satisfying the following:

(1) If a sequence of points of A converges in S, then there exists a Cauchy subsequence under d.

(2) If a Cauchy sequence of points of A under d has a convergent subsequence in S, then it converges in S.

His proof to the above result also contains the characterization of closed images of complete metric spaces which are called briefly complete Lašnev. The object of this note is to report it.

THEOREM 2. A space S is complete Lašnev iff S is a regular Fréchet space with a dense, G_δ , metrizable subspace (A, d) satisfying (1) and (2)′:

(2)′ If a sequence of points of A is Cauchy under d, then it converges in S.

PROOF. The necessity: Let f be a closed mapping of a complete metric space (M, d) onto S. By [2, Th. 55.12] we can assume that f is irreducible. Consider the decomposition space Ⓗ = {$f^{-1}(p)$:

$p \in S\}$ of M, which is homeomorphic to S. Let

$$(H)_1 = \{K \in (H) : \text{diam}(k); \text{ not defined or} \geqq 1\}.$$

For each $n > 1$, put

$$(H)_n = (H)_1 \cup \{K \in (H) : \text{diam}(K) \geqq \frac{1}{n}\}.$$

If $p \in M - (H)_n^{\#}$ ($(U)^{\#}$ denotes the union of all members of (U)), then $p \in K \in (H)$ with $\text{diam}(K) < 1/n$.

$$R = \{x \in M : d(K,x) < \frac{1}{4}(\frac{1}{n} - \text{diam}(K))\}$$

is an open set such that $K \subset R$ and $\text{diam}(R) < 1/n$. Then since f is closed, $\{K \in (H) : K \subset R\}^{\#}$ is an open neighborhood(=nbd) of p, which is disjoint from $(H)_n^{\#}$. Therefore $M - (H)_n^{\#}$ is an open set of M. It follows from the irreducibility of f that $M - (H)_n^{\#}$ is dense in M. By [2, Th. 7.11]

$$A = \bigcap_n (M - (H)_n^{\#})$$

is dense in M. Observe that

$$A = \{f^{-1}(p) : |f^{-1}(p)| = 1, p \in S\}.$$

Since A is considered as the subspace of (M, d), A is metrizable. It is easily seen that (A, d) satisfies (1) and (2)$'$.

The sufficiency: Let S be a regular Fréchet space with a dense metrizable subspace (A, d) satisfying (1) and (2)$'$. Let M be the usual completion of a metric space (A, d). Then we define a mapping $f : M \to S$ as follows: If $p \in A$, then $f(p) = p$ and if $p \in M - A$, then $f(p) = x$, where for $\{p_n\} \subset A$

$$x = \lim_{n\to\infty} p_n \text{ in } S \quad \text{and} \quad \lim_{n\to\infty} p_n = p \text{ in } M.$$

It is easily verified by(1) and (2)$'$ that f is a well-defined mapping of M onto S. To see the continuity of f, let $\{x_n\}$ be a sequence of points of M such that $x_n \to x$ in M as $n \to \infty$. The we shall show that $f(x_n) \to f(x)$ in S. Suppose the contrary. Then there exists an open nbd V' of f(x) which does not contain the infinite subsequence $\{f(x_n')\}$ of $\{f(x_n)\}$. Since S is regular

$$x \in V \subset \overline{V} \subset V'$$

for some open set V. For each i, choose an open nbd R_i of $f(x_i')$ such that $R_i \cap \overline{V} = \phi$. Since for each i there exists a sequence $\{x_{ij} : j=1,2,\dots\}$ of points of A which converges to x_i in M and to $f(x_i)$ in S, we can choose a sequence $\{y_i\}$ of A such that

$$y_i \in R_i \text{ and } d(x_i, y_i) < \frac{1}{i}.$$

Since $y_i \to x$ and consequently $y_i \to f(x)$ in S by (2)', this is a contradiction. This proves the continuity of f. To see the closedness of f, let F be an arbitrary closed set of M. Suppose $y \in \overline{f(F)} - f(F)$. Since S is Fréchet, $y = \lim_{i\to\infty} f(x_i)$ for a sequence $\{x_i\}$ of points of F. We can assume without loss of generality that

$$\{x_i\} \cap A = \phi \ , \ \{x_i\} \cap f^{-1}(y) = \phi \ \text{ and } \ x_i \neq x_j \ (i \neq j).$$

Let $\{x_{ij} : j=1, 2, \dots\}$ be a sequence in A such that $x_{ij} \to x_i$ in M, $\to f(x_i)$ in S and $d(x_i, x_{ij}) < 1/ij$. Since S is Fréchet, $y = \lim_{n\to\infty} y_n$, where $\{y_n\} \subset \{x_{ij} : i, j = 1, 2, \dots\}$ and

$$|\{y_n\} \cap \{x_{nj} : j=1, 2, \dots\}| \leqq 1$$

for each n. By (1) there exists a Cauchy subsequence $\{x_{n(c)m(c)} : c=1, 2, \dots\}$ of $\{y_n\}$, where $n(c), m(c) \geqq c$ for each c. It follows easily that

$$x_{n(1)}, x_{n(1)m(1)}, x_{n(2)}, x_{n(2)m(2)}, \dots$$

is Cauchy under d. Since M is complete, $x_{n(c)m(c)} \to p \in f^{-1}(y)$. Since p is a cluster point of $\{x_i\}$, $p \in F$, i.e. $f(p) \in f(F)$. This is a contradiction. This completes the proof.

With respect to the property of complete Lašnev spaces, Van Doren gave in [1] the following:

THEOREM 3. (Van Doren). A complete Lasnev space contains a dense subspace which is metrizable in a complete manner.

The proof of the necessity of Th. 2 is another one to this result.

REFERENCES

[1] K.R. Van Doren, Closed, continuous images of complete metric spaces, Fund. Math. 80(1973) 47-50.

[2] Y. Kodama and K. Nagami, Theory of topological spaces, Iwanami (Tokyo), 1974.

[3] S. A. Stricklen Jr., An embedding theorem for Lašnev spaces, General Top. Appl. 6(1976) 153-165.