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Multidimensional Central and Local Limit Theorems for the Phase Separation Line in the Two-Dimensional Ising Model

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§1. The Two-Dimensional Ising Model

Let $\mathbb{Z}^2$ be the square lattice and L be its dual lattice; *i.e.*

$$\mathbb{Z}^2 \equiv \{(x_1,\ x_2);\ x_1 \text{ and } x_2 \text{ are integers}\},$$

$$L \equiv \{(x_1,\ x_2) + (\tfrac{1}{2},\ \tfrac{1}{2});\ (x_1,\ x_2) \in \mathbb{Z}^2\}.$$

We consider $\mathbb{Z}^2$ and L also as graphs. Let $\Omega \equiv \{-1,\ +1\}^L$ be the space of possible spin configurations on L. For each positive integer N, we define a square box V_N by

$$V_N \equiv \{(x_1,\ x_2) + (\tfrac{1}{2},\ \tfrac{1}{2});\ (x_1,\ x_2) \in \mathbb{Z}^2,\ 0 \le x_1 \le N-1,\ -[\tfrac{N}{2}] \le x_2 \le [\tfrac{N-1}{2}]\},$$

where $[u]$ is the largest integer smaller than u. For given $\omega \in \Omega$ and $\beta > 0$, the finite Gibbs state on $\Omega_N \equiv \{-1,\ +1\}^{V_N}$ with boundary condition ω, inverse temperature β (in the absence of the exterior magnetic field) is given by

$$P_N^{\omega}(\eta) = Z_N(\omega)^{-1} \exp\{-\beta\ U_N(\eta:\ \omega)\} \qquad \eta \in \Omega_N$$

where $Z_N(\omega) \equiv \sum_{\eta' \in \Omega_N} \exp\{-\beta\ U_N(\eta':\ \omega)\}$, and

$$U_N(\eta;\ \omega) \equiv -\sum_{x,y \in V_N}^{*} \eta(x)\eta(y) - \sum_{\substack{x \in V_N \\ y \in \partial V_N}}^{*} \eta(x)\omega(y),$$

where the summation $\sum_{x \in A,\, y \in B}^{*}$ is taken over all pairs $(x,\ y)$ such that (i) x and y are nearest neighbours in L, and (ii) $x \in A$ and $y \in B$. From now on, we fix the boundary condition ω as

$$\omega\big((x_1,\ x_2) + (\tfrac{1}{2},\ \tfrac{1}{2})\big) = \begin{cases} +1 & \text{if} \quad x_2 \ge 0 \\ -1 & \text{if} \quad x_2 < 0, \end{cases}$$

and we write simply P_N instead of P_N^{ω}.

§2. The Phase Separation Line

Let us fix $N > 0$ arbitrarily. Define

$$\tilde{V}_N \equiv \{(x_1, x_2) \in \mathbb{Z}^2;\ 0 \le x_1 \le N,\ -[\tfrac{N}{2}] \le x_2 \le [\tfrac{N+1}{2}]\}.$$

We can regard $\tilde{V}_N$ as a subgraph of $\mathbb{Z}^2$. We call a segment of length one connecting two points which are nearest neighbours in $\mathbb{Z}^2$ (or in L) by "a bond in $\mathbb{Z}^2$(or in L)". Then for each $\eta \in \Omega_N$, we can define a subgraph $C_N(\eta)$ of $\tilde{V}_N$ in the following way. A bond in $\mathbb{Z}^2$ belongs to $C_N(\eta)$ if and only if it crosses a bond in L connecting two points $x, y \in L$ such that $\eta(x)\eta(y) = -1$ $\left(\text{or } \eta(x)\omega(y) = -1\right)$. $C_N(\eta)$ consists of some connected components, and each vertex of $C_N(\eta)$ belongs to two of four bonds of $C_N(\eta)$ unless the vertex is $A = (0, 0)$ or $B = (N, 0)$. It is easy to see that the connected component of $C_N(\eta)$ containing A also contains B. We denote this component by $\lambda_N(\eta)$, and call it by "the phase separation line". Let $\Lambda_N \equiv \{\lambda_N(\eta) : \eta \in \Omega_N\}$, and $S \equiv \{\gamma;$ subgraph of $\mathbb{Z}^2$ such that (i) the length $|\gamma|$ is finite, and (ii) each vertex of γ belongs to two or four bonds of $\gamma\}$.

Theorem (Gallavotti)

There exists a function $\varphi : \mathcal{N} = \bigoplus_{n=0}^{\infty} S^n \to \mathbb{R}$ such that

(i) φ is symmetric, (ii) $\sum_{\substack{\Gamma \in \mathcal{N} \\ \Gamma \ni 0}} |\varphi(\Gamma)| < +\infty$ and (iii) for each $\lambda \in \Lambda_N$,

$$P_N\left(\{\eta \in \Omega_N;\ \lambda_N(\eta) = \lambda\}\right)$$

$$= \exp\{-2\beta|\lambda| - \sum_{\substack{\Gamma \in \mathcal{N} \\ \Gamma \cap \lambda \neq \phi \\ \Gamma \subset V_N}} \varphi(\Gamma)\} \Big/ \sum_{\lambda' \in \Lambda_N} \exp\{-2\beta|\lambda'| - \sum_{\substack{\Gamma \in \mathcal{N} \\ \Gamma \cap \lambda' \neq \phi \\ \Gamma \subset \tilde{V}_N}} \varphi(\Gamma)\}.$$

We denote the left hand side of the above equality simply by $P_N(\lambda)$. Thus we are given a sequence of probability spaces (Λ_N, P_N). To state our result, we need some notations. For each $\lambda \in \Lambda_N$, $0 \le \ell \le N$, let $\overline{X}_N(\ell)$ and $\underline{X}_N(\ell)$ be $\overline{X}_N(\ell) \equiv \max\{k;(k, \ell) \in \lambda\}$, $\underline{X}_N(\ell) \equiv \min\{k; (k, \ell) \in \lambda\}$.

Theorem 1.

Let an integer $k \ge 1$, and $0 < t_1 < t_2 < \dots < t_k < 1$ and the numbers $-\infty < T_j < T_j' < \infty$, $j = 1, 2, \dots, k$ be given arbitrarily. Let $a_j^{(N)} \equiv [t_j \cdot N]$ $j = 1, 2, \dots, k$, where $[u]$ is the largest integer smaller than u. Then,

$$\lim_{N\to\infty} P_N\left(\bigcap_{j=1}^{k} \{T_j \le \frac{\overline{X}_N(a_j^{(N)})}{\sigma\sqrt{N}} \le T_j'\}\right)$$

$$= \lim_{N\to\infty} P_N\left(\bigcap_{j=1}^{k} \{T_j \le \frac{\underline{X}_N(a_j^{(N)})}{\sigma\sqrt{N}} \le \frac{\overline{X}_N(a_j^{(N)})}{\sigma\sqrt{N}} \le T_j'\}\right)$$

$$= P_{0;0}^{1;0}\left(X_{t_j} \in [T_j, T_j'],\ j = 1, 2, \cdots, k\right)$$

where $\sigma = \sigma(\beta) > 0$ is a constant depending only on $\beta > 0$ and $(\{X_t\}_{0 \le t \le 1}, P_{0;0}^{1;0})$ is a one-dimensional Brownian Bridge such that $P_{0;0}^{1;0}(X_0 = X_1 = 0) = 1$.

Remark Gallavotti has proved the above theorem in the case when $k = 1$ in [2], and it is announced that Cammarota has proved it for general $k \ge 1$, but we have got nothing in print yet.

§3. An Auxiliary Ensemble and the Central Limit Theorem

In order to prove Theorem 1, we need another sequence of probability spaces $(\hat{\Lambda}_N, \hat{P}_N)$. Let $I_N \equiv \{(x_1, x_2) \in {}^2; 0 \le x_1 \le N\}$, and $\hat{\Lambda}_N \equiv$ {connected subgraph λ of I_N such that (i) $|\lambda| < +\infty$, (ii) $\lambda \ni A$, (iii) there exists a point B' in $\{(N, \ell) : \ell \in \mathbb{Z}\}$

such that each vertex of λ belongs to two or four bonds of λ unless it is A or B', and A and B' belong to one or three bonds of $\lambda\}$. For each $\lambda \in \hat{\Lambda}_N$, we define

$$\hat{P}_N(\lambda) \equiv \exp\{-2\beta|\lambda| - \sum_{\substack{\Gamma\in\mathcal{N} \\ \Gamma\subset I_N \\ \Gamma\cap\lambda\neq\phi}} \varphi(\Gamma)\} \Big/ \sum_{\tilde{\lambda}\in\hat{\Lambda}_N} \exp\{-2\beta|\tilde{\lambda}| - \sum_{\substack{\Gamma\in\mathcal{N} \\ \Gamma\subset I_N \\ \Gamma\cap\tilde{\lambda}\neq\phi}} \varphi(\Gamma)\}.$$

Then, Gallavotti has shown that for every $A_N \subset \Lambda_N$,

$$P_N(A_N) = \hat{P}_N(A_N \mid \tilde{\Lambda}_N) + o(\frac{1}{N^{\ell}}) \qquad \text{as} \quad N \to \infty \quad \forall \ell \in \mathbb{N},$$

where $\tilde{\Lambda}_N \equiv \{\lambda \in \hat{\Lambda}_N;\ B' = B\}$.

Hence we only have to investigate the probability space $(\hat{\Lambda}_N, \hat{P}_N)$. we obtain the following theorem first, which is much easier to prove than Theorem 1.

Theorem 2.

For each integer $k \geq 1$, $0 = t_0 < t_1 < \cdots < t_k < t_{k+1} = 1$ and $-\infty < T_j < T_j' < +\infty$, $j = 1, 2, \ldots, k+1$, we have

$$\lim_{N\to\infty} \hat{P}_N\left(\bigcap_{j=1}^{k+1}\{T_j \leq \frac{\overline{X}_N(a_j^{(N)})}{\sigma\sqrt{N}} \leq T_j'\}\right)$$

$$= \lim_{N\to\infty} \hat{P}_N\left(\bigcap_{j=1}^{k+1}\{T_j \leq \frac{\underline{X}_N(a_j^{(N)})}{\sigma\sqrt{N}} \leq \frac{\overline{X}_N(a_j^{(N)})}{\sigma\sqrt{N}} \leq T_j'\}\right)$$

$$= P_0\left(X_{t_j} \in [T_j, T_j'], j = 1, 2, \ldots, k+1\right),$$

where $(\{X_t\}, P_0)$ is a one-dimensional Brownian Motion starting at 0.

References

[1] Del Grosso, G., On the Local Central Limit Theorem for Gibbs Processes. Comm. Math. Phys., 37, 1974.

[2] Gallavotti, G., The Phase Separation Line in the Two-Dimensional Ising Model. Comm. Math. Phys., 27, 1972.

[3] Gallavotti, G. and Martin-Löf, A., Surface Tension in the Ising Model. Comm. Math. Phys., 25, 1972.