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Hyperbolic nonwandering sets without dense periodic points

by

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Let $f: M \longrightarrow M$ be a C^∞ diffeomorphism of a closed C^∞ manifold M, and let $\Omega(f)$ be the nonwandering set of f. $\Omega(f)$ is hyperbolic if $\Omega(f)$ is compact and the restriction $T_{\Omega(f)}M$ of the tangent bundle TM of M on $\Omega(f)$ splits into the Whitney sum of Tf-invariant subbundles

$$T_{\Omega(f)}M = E^s \oplus E^u,$$

such that given a Riemannian metric on TM there are positive numbers c and $\lambda < 1$ such that $|Tf^n v| < c\lambda^n |v|$, for $v \in E^s$ and $n > 0$, and $|Tf^{-n} v| < c\lambda^n |v|$, for $v \in E^u$ and $n > 0$. The following problem was suggested in [3].

Problem. If a nonwandering set $\Omega(f)$ is hyperbolic, are the periodic points dense in $\Omega(f)$?

Newhouse and Palis proved that the answer is affirmative when M is a two dimensional closed manifold ([1], [2]).

In this paper we give the following.

Theorem. Suppose $\dim M \geq 4$. Then there is a diffeomorphism $F: M \longrightarrow M$ such that the nonwandering set $\Omega(F)$ is hyperbolic but its periodic points are not dense in $\Omega(F)$.

Construction.

To simplify the construction, we assume $\dim M = 4$.

1. Denote $D = [-2, 6] \times [-1, 3] \subset R^2$. Let an embedding $f: D \longrightarrow D$ satisfy the followings (figure 1). Suppose that real numbers $a_{-1}, \cdots, a_6$ satisfy

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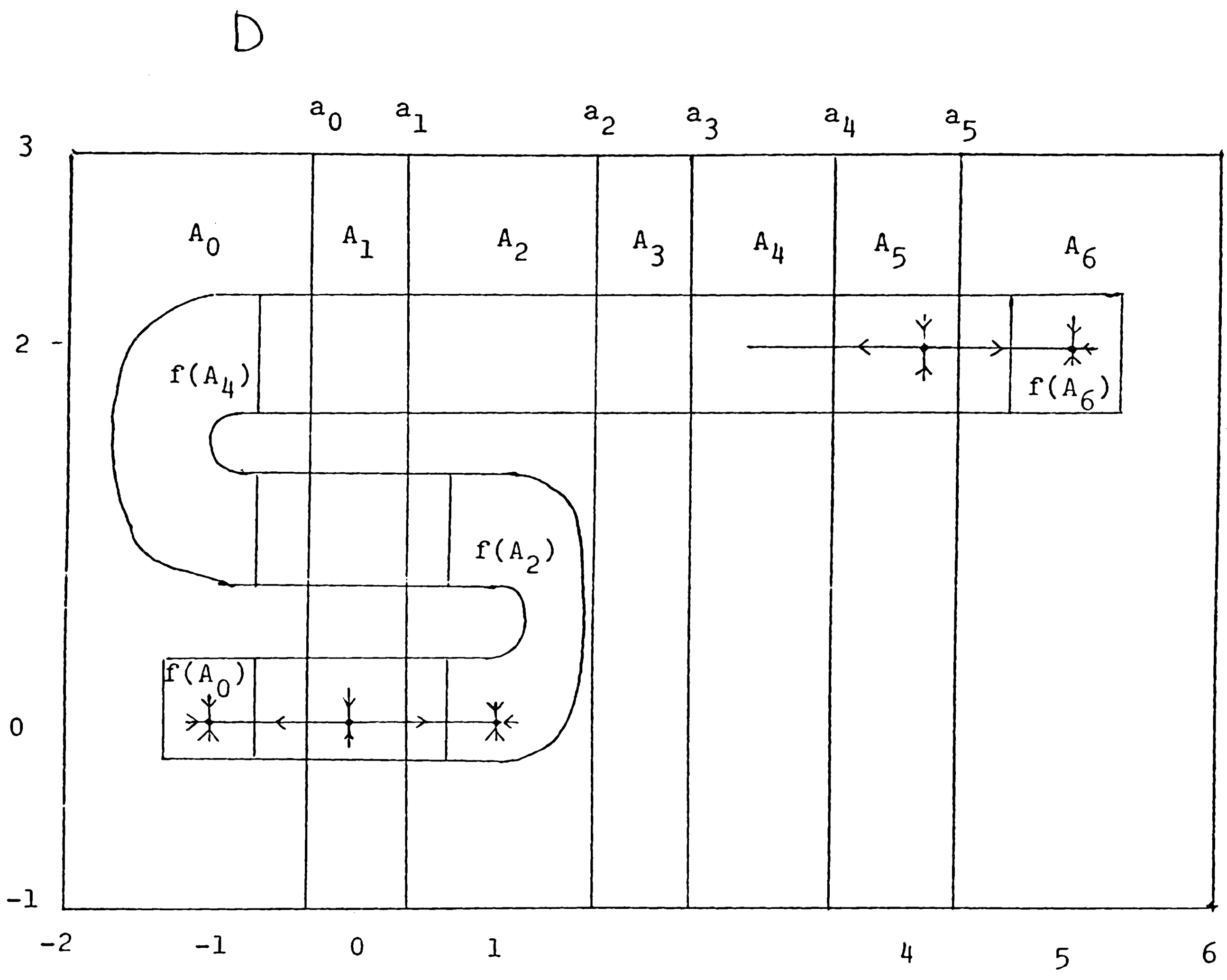
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f(A2)
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3
2
0
-1
-2
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1
4
5
6

figure 1

<u>1.1</u> $a_{-1} = -2 < -1 < a_0 = -a_1 < 0 < a_1 < 1 < a_2 < a_3 < a_4 < 4 < a_5 < 5 < a_6 = 6$,

and the rectangle A_i ($i = 0,\cdots,6$) is given by

$$A_i = \{ (x,y) \in D \mid a_{i-1} \le x \le a_i \}.$$

Then f satisfies <u>1.2</u> $\sim$ <u>1.5</u>.

<u>1.2</u> $f|A_0$, $f|A_2$ and $f|A_6$ are contractions with three sinks (-1,0), (1,0) and (5,2),

<u>1.3</u> $f(A_4) \subset \mathrm{int}A_0$,

<u>1.4</u> $f|A_i : A_i \longrightarrow f(A_i)$ ($i = 1,3,5$) maps A_i linearly onto a rectangle $f(A_i)$, expanding horizontally and contracting vertically. There are two hyperbolic fixed points, (0,0) and (4,2).

<u>1.5</u> There are numbers $\alpha > 1$ and $0 < \beta < 1$ such that

$$f(x,y) = \begin{cases} (\alpha x, \beta y) & \text{for } (x,y) \in A_1 \\ (\alpha(x-4)+4, \beta(y-2)+2) & \text{for } (x,y) \in A_5. \end{cases}$$

2. Let $D' \subset R^2$ satisfy the followings (figure 2). D' is a neighbourhood of $(\{0\} \times [-1, 1]) \cup ([-2, 0] \times \{0\})$ which is diffeomorphic to a 2-dimensional disk, and there is a sufficiently small positive number ε such that

$$\{(x,y) \in D' \mid |y+1| \le \varepsilon\} = [-\varepsilon, \varepsilon] \times [-1-\varepsilon, -1+\varepsilon]$$

and

$$\{(x,y) \in D' \mid |x+1| \le \varepsilon\} = [-1-\varepsilon, -1+\varepsilon] \times [-\varepsilon, \varepsilon].$$

Let an embedding $g: D' \longrightarrow D'$ satisfy <u>2.1</u> $\sim$ <u>2.9</u>.

<u>2.1</u> $g(D') \subset \mathrm{int}D'$,

<u>2.2</u> g is isotopic to the identity,

<u>2.3</u> $\bigcap_{n>0} g^n(D') = (\{0\} \times [-1, 1]) \cup ([-2, 0] \times \{0\})$,

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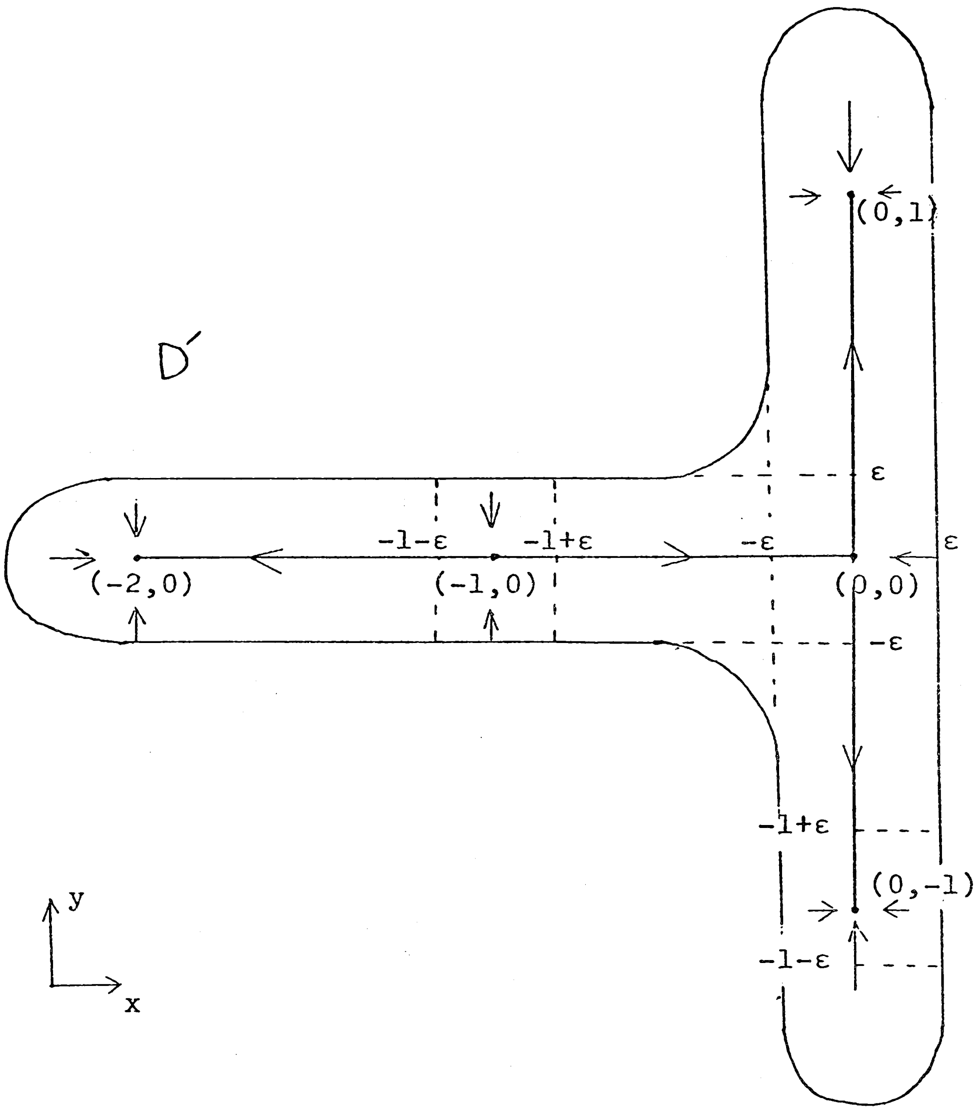

figure 2

2.4 There are five fixed points: three sinks $(-2,0)$, $(0,1)$, $(o,-1)$, and two saddle points $(0,0)$, $(-1,0)$.

2.5 $W^u((0,0)) = \{0\} \times (-1,1)$,

2.6 $W^u((-1,0)) = (-2,0) \times \{0\}$,

2.7 $W^s((0,0)) \cap D' = \{(x,0) \in D' \mid x \geq -1\}$,

where $W^s(p)$ (resp. $W^u(p)$) is the stable (resp. unstable) manifold through p. $(-1,1)$ and $(-2,0)$ denote open intervals.

2.8 $g(x,y) = (\frac{1}{2}x, \frac{1}{2}(y+1)-1)$ if $|y+1| \leq \varepsilon$,

2.9 $g(x,y) = (2(x+1)-1, \frac{1}{2}y)$ if $|x+1| \leq \varepsilon$.

3. Define

$$N = D \times D' \cup_\psi D^3(\delta) \times [0, 1],$$

where

$$D^3(\delta) = \{(y_1,y_2,y_3) \in R^3 \mid \sqrt{y_1{}^2+y_2{}^2+y_3{}^2} \leq \delta \}$$

and

$$0 < \delta < \tfrac{1}{4}\varepsilon.$$

The attaching map

$$\psi : D^3(\delta) \times ([0, \varepsilon] \cup [1-\varepsilon, 1]) \longrightarrow D \times D'$$

is given by

$$\psi(y_1,y_2,y_3,t) = \begin{cases} (y_1,y_2,t,y_3-1) & \text{if } 0 \leq t \leq \varepsilon \\ (y_1+4,y_2+2,y_3-1,1-t) & \text{if } 1-\varepsilon \leq t \leq 1 \end{cases}$$

(figure 3).

In 4 $\sim$ 10, we will construct an embedding $F:N \longrightarrow N$. After this, (x_1,x_2,x_3,x_4) (resp. (y_1,y_2,y_3,t)) denotes a point of $D \times D' \subset N$ (resp. $D^3(\delta) \times [0, 1] \subset N$).

4. For $(x_1,x_2,x_3,x_4) \in D \times D'$ with $|x_3+1| \geq \varepsilon$ and $|x_4+1| \geq \varepsilon$, define

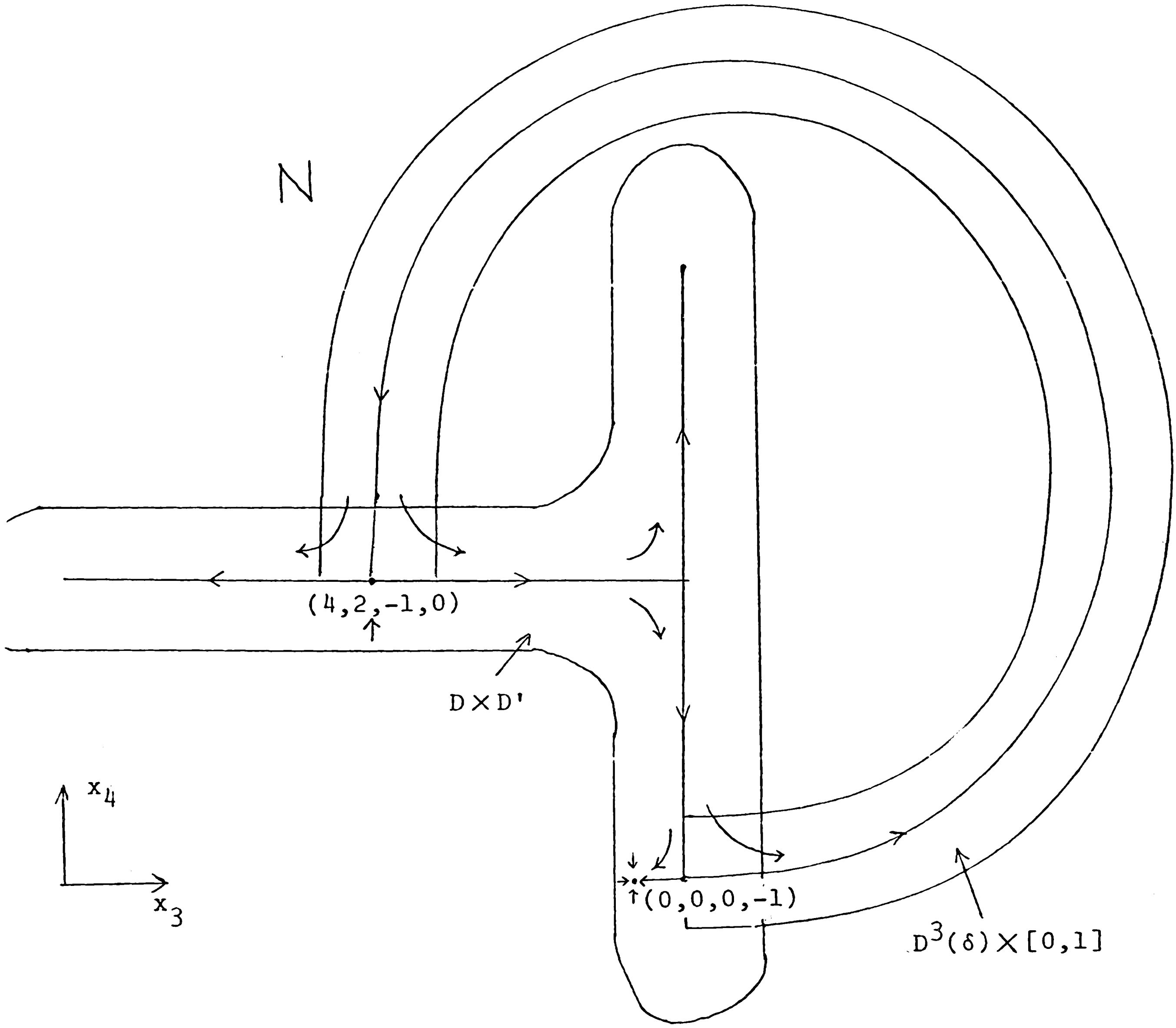

figure 3

4.1 $F(x_1,x_2,x_3,x_4) = (f(x_1,x_2),g(x_3,x_4))$.

5. For $(x_1,x_2,x_3,x_4) \in D \times D'$ with $\frac{1}{4}\varepsilon \le |x_4+1| \le \varepsilon$, define

5.1 $F(x_1,x_2,x_3,x_4) = (f_{|x_4+1|}(x_1,x_2),\ g(x_3,x_4))$, where $f_t : D \longrightarrow D$ $(0 \le t \le \varepsilon)$ is an isotopy satisfying 5.2 ~ 5.6.

Suppose that positive numbers $b_1,\cdots,b_4$ satisfy

5.2 $0 < b_1 < b_2 < \delta < b_3 < b_4 < a_1$,

$\alpha b_1 < b_2$,

and

$b_4 < \min\{\ 4-a_4,\ a_5-4\ \}$.

Then

5.3 $f_t(x_1,x_2) = f(x_1,x_2)$ if $|x_1| < b_1$ or $|x_1| > b_4$,

5.4 $f_t = f$ for $\frac{1}{2}\varepsilon \le t \le \varepsilon$,

5.5 $f_t = f_0$ for $0 \le t \le \frac{1}{4}\varepsilon$,

and

5.6 $f_t(x_1,x_2) = (\bar{f}_t(x_1),\ \beta x_2)$ for $|x_1| \le b_4$,

where $\bar{f}_t$ is an isotopy of a neighbourhood of 0 in R^1 and $\bar{f}_0$ has five fixed points: three sources 0, $\pm b_3$, and two sinks $\pm b_2$.

6. For $(x_1,x_2,x_3,x_4) \in D \times D'$ with $|x_4+1| < \frac{1}{4}\varepsilon$, F is defined as follows. Let

6.1 $U = \{(x_1,x_2,x_3,x_4) \in D \times D' \mid \sqrt{x_1^2+x_2^2+(x_4+1)^2} \le \delta\}$,

and

6.2 $U_1 = \{(x_1,x_2,x_3,x_4) \in D \times D' \mid \sqrt{x_1^2+x_2^2+(x_4+1)^2} \le \delta_1\}$,

where $b_2 < \delta_1 < \delta$.

Then F is defined as follows.

<u>6.3</u> $F(x_1,x_2,x_3,x_4) = (f_0(x_1,x_2), g(x_3,x_4))$

if $(x_1,x_2,x_3,x_4) \in D \times D' - U$ and $|x_4+1| < \frac{1}{4}\varepsilon$,

<u>6.4</u> $F(x_1,x_2,x_3,x_4) = (f_0(x_1,x_2), \bar{g}(x_1,x_2,x_3,x_4), \frac{1}{2}(x_4+1)-1)$

if $(x_1,x_2,x_3,x_4) \in U \cap F^{-1}(U)$,

where $\bar{g}$ satisfies <u>6.5</u> $\sim$ <u>6.7</u>.

<u>6.5</u> $\bar{g}(x_1,x_2,x_3,x_4) = \frac{1}{2}x_3$ near the frontier of U,

<u>6.6</u> $\bar{g}(x_1,x_2,x_3,x_4) = 2x_3$

if $(x_1,x_2,x_3,x_4) \in U_1$ and $-\frac{1}{4}\varepsilon \leq x_3 \leq \frac{1}{2}\varepsilon$,

and

<u>6.7</u> $\bar{g}(x_1,x_2,x_3,x_4)$ does not depend on x_1 if $|x_1| < b_1$.

<u>6.8</u> $F(\{(x_1,x_2,x_3,x_4) \in U \mid x_3 < 0\})$

$\subset \{(x_1,x_2,x_3,x_4) \in U \mid x_3 < 0\}$.

In $\{(x_1,x_2,x_3,x_4) \in U \mid x_3 < 0\}$ there are only a finite number of nonwandering points, which are hyperbolic fixed points. Furthermore F satisfies the conditions in 10.

7. On $D^3(\delta) \times [0, 1-\varepsilon]$, F is given as follows

<u>7.1</u> $F(y_1,y_2,y_3,t) = (f_0(y_1,y_2), \frac{1}{2}y_3, \phi(y_1,y_2,y_3,t)) \in D^3(\delta) \times [0,1]$,

where ϕ satisfies the followings.

If $\sqrt{{y_1}^2+{y_2}^2+{y_3}^2} < \delta_1$ or $\frac{1}{2} < t$,

<u>7.2</u> $\phi(y_1,y_2,y_3,t)$ depends only on t

and

<u>7.3</u> $\frac{\partial\phi}{\partial t} > 0$.

<u>7.4</u> $\phi(y_1,y_2,y_3,t) = 1 - \frac{1}{2}(1-t)$ for $1-2\varepsilon \leq t \leq 1-\varepsilon$.

<u>7.5</u> $\phi(y_1,y_2,y_3,t) = \bar{g}(y_1,y_2,t,y_3-1)$ if $0 \leq t \leq \varepsilon$.

Moreover F satisfies 10.

8. For $(x_1,x_2,x_3,x_4) \in D \times D'$ with $|x_3+1| < \frac{1}{4}\varepsilon$,

F is given as follows. Let $h_t : D \longrightarrow D$ ($0 \le t \le \varepsilon$) be an isotopy such that

8.1 $h_t = f$ if $\frac{1}{2}\varepsilon \le t \le \varepsilon$,

8.2 $h_t(x_1,x_2) = f(x_1,x_2)$

if $-2 \le x_1 \le 4-b_4$ or $4+b_4 \le x_1 \le 6$,

and

8.3 $h_t(x_1,x_2) = f(x_1-4,x_2-2)+(4,2)$ if $|x_1-4| < b_4$.

Then

8.4 $F(x_1,x_2,x_3,x_4) = (h_0(x_1,x_2),\bar{h}(x_1,x_2,x_3,x_4),\frac{1}{2}x_4)$,

where $\bar{h}$ satisfies the followings.

8.5 $\bar{h}(x_1,x_2,x_3,x_4) = \frac{1}{2}(x_3+1)-1$

if $\sqrt{(x_1-4)^2+(x_2-2)^2+(x_3+1)^2} \le \delta$ and $x_4 > \frac{2}{3}\varepsilon$,

8.6 $\bar{h}(x_1,x_2,x_3,x_4) = 2(x_3+1)-1$

if $\sqrt{(x_1-4)^2+(x_2-2)^2+(x_3+1)^2} \ge \delta_2$ or $x_4 < \frac{1}{3}\varepsilon$,

where $\delta < \delta_2 < \frac{1}{4}\varepsilon$.

8.7 $\bar{h}(x_1,x_2,x_3,x_4)$ does not depend on x_1 if $|x_1-4| < b_1$.

Furthermore F satisfies 10.

9. For $(x_1,x_2,x_3,x_4) \in D \times D'$ with $\frac{1}{4}\varepsilon \le |x_3+1| < \varepsilon$, define

9.1 $F(x_1,x_2,x_3,x_4) = (h_{|x_3+1|}(x_1,x_2),2(x_3+1)-1,\frac{1}{2}x_4)$.

10. F is an embedding of N such that

10.1 $F(N) \subset \operatorname{int} N$,

and

10.2 F is isotopic to the identity.

11. Straightening the corner (and modifying F near the

corner), we can regard N as a submanifold of M which is diffeomorphic to $D^3 \times S^1$. Extend F to a diffeomorphism of M such that the nonwandering sets of F in M-N consists of a finite number of hyperbolic fixed points.

12. The nonwandering set of F consists of a finite number of hyperbolic fixed points and two non-periodic orbits $\{(x_1,x_2,0,0) \in D \times D' \mid (x_1,x_2) \text{ satisfies } 12,i\}$ (i = 1,2), where

<u>12.1</u> there is an integer n_0 such that

$$f^n(x_1,x_2) \in A_5 \quad \text{if} \quad n < n_0,$$
$$f^n(x_1,x_2) \in A_3 \quad \text{if} \quad n = n_0,$$
$$f_n(x_1,x_2) \in A_1 \quad \text{if} \quad n > n_0,$$

and

<u>12.2</u> there is an integer n_0 such that

$$f^n(x_1,x_2) \in A_5 \quad \text{if} \quad n < n_0,$$
$$f^n(x_1,x_2) \in A_1 \quad \text{if} \quad n \geq n_0.$$

The details will be published elsewhere.

References

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[3] S.Smale, Differentiable dynamical systems, Bull. Amer. Math. Soc., 73 (1967), 747-817.