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A WEAK EQUIVALENCE AND TOPOLOGICAL ENTROPY .

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We will investigate the topological entropies of weakly equivalent topological flows. Let X be a compact metric space. A topological flow $\Phi = \{\phi_t \ ; -\infty<t<\infty\}$ on X is a one parameter group of homeomorphisms of X which is continuous in (t,x). Let Φ and Ψ be flows on X and Y respectively. They are weakly equivalet by definition if there is a homeomorphism π of X onto Y such that $\{\pi^{-1}\circ \psi_t \circ \pi(x) \ ; t\in R\} = \{\phi_t(x) \ ; t\in R\}$. It is a generalization of time changes of flows given by [5]. In [5] Totoki investigated the changes of metrical entropy of metrical flows under time changes. Time changes by integrable additive functionals preserve the properties that the entropy is zero, positive finite or infinite respectively. Our first result is analogus to the above one. To simplify we assme that X = Y .

<u>Theorem 1.</u> Φ and Ψ are weakly equivalent and they have no fixed point, then we have $c_1 h(\psi_1) \leq h(\phi_1) \leq c_2 h(\psi_1)$ where c_1, c_2 are positive constants of ϕ_1, ψ_1.

In order to prove the theorem we will apply measure theoritical method. Let us denote by $\mathcal{E}(\Phi)$ ($\mathcal{E}(\phi_1)$) the set of all $\Phi(\phi_1)$-invariant ergodic probability measures. Let $h_\mu(\phi_1)$ be the metrical entropy of ϕ_1 for $\mu\in\mathcal{E}(\phi_1)$.

<u>Lemma 1.</u> ([6])

$$h(\phi_1) = \sup_{\mu\in\mathcal{E}(\phi_1)} h_\mu(\phi_1) = \sup_{m\in\mathcal{E}(\Phi)} h_m(\phi_1).$$

Under the assumption of the theorom we can and do assume further that and have the same orbits with the same directions. Then we can define $\theta(t,x)$, $-\infty < t < \infty$, $x \in X$ with the following properties (cf.[2]) ;

i) $\phi_t(x) = \psi_{\theta(t,x)}(x)$

ii) $\theta(t+s,x) = \theta(t,\phi_s(x)) + \theta(s,x)$

iii) $\theta(0,x) = 0$ and $\theta(t,x)$ is strictly increasing

iv) $\theta(t,x)$ is continulus in (t,x).

<u>Lemma 2. ([5])</u>

For $m \in \mathcal{E}(\Phi)$ if $E_m(\theta(1,x))$ then

$$h_{\hat{m}}(\psi_1) = \frac{1}{E_m(\theta(1,x))} h_m(\phi_1).$$

<u>Lemma 3.</u>

For $m \in \mathcal{E}(\Phi)$,

$$E_{\hat{m}}(f) = \frac{1}{E_m(\theta(1,x))} E_m\left(\int_0^{\theta(1,x)} f(\psi_t(X))dt\right)$$

define an $\hat{m} \in \mathcal{E}(\Psi)$. The map $\wedge : m \in \mathcal{E}(\Phi) \longrightarrow m \in \mathcal{E}(\Psi)$ is bijective and $G(\Phi,m) = G(\Psi,\hat{m})$

where $G(\Phi,m) = \{x \in X ;\ \lim_{T\to\infty} \frac{1}{T} \int_0^T f(\phi_t(x))dt = E_m(f)$

for all continuous f .

Corollary 1. Any time changed (weakly equivalent) flow of a strictly ergodic (i,e. minimal and uniquely ergokic) flow is also strictly ergodic.

Now combining Lemma 1,2 and 3, we can prove the theorem. We have

$$h(\phi_1) = \sup_{m\in \mathcal{E}(\Phi)} h_m(\phi_1) = \sup_{m\in \mathcal{E}(\Phi)} E_m(\theta(1,x))\, h_{\hat{m}}(\psi_1),$$

and so putting $c_1 = \min_{x\in X} \theta(1,x)$ and $c_2 = \max_{x\in X} \theta(1,x)$

$$c_1 h(\psi_1) = c_1 \sup_{\hat{m}\in\mathcal{E}(\Psi)} h_{\hat{m}}(\psi_1) = c_1 \sup_{m\in\mathcal{E}(\Phi)} h_{\hat{m}}(\psi_1) \leq h(\Phi)$$

$$\leq c_2 \sup_{m\in\mathcal{E}(\Phi)} h_{\hat{m}}(\psi_1) = c_2\, h(\psi_1).$$

Next we will find that if Φ and Ψ have fixed point $\{x_0\}$ then the result of theorem 1 does not hold. In fact we get a stronger result in the view point of invariant measures.

Theorem 2. There exists a pair of weakly equivalent flows Φ and Ψ with fixed point $\{x_0\}$ such that

1) there exists a $\mu \in \mathcal{E}(\Phi)$ such that $0 < h_\mu(\phi_1) < \infty$,

2) there is no non-trivial Ψ-invariant ergodic probability measure, that is $\mathcal{E}(\Phi) = \{\delta_{x_0}\}$.

Corollary 2. For Φ and Ψ in Theorem 2, we have $h(\psi_1) = 0$ and $0 < h(\phi_1)$

An example of the pair Φ and Ψ is constructed as the flows under the functions (the special flows or suspensions) . The essential point is that the integrability of $\theta(1,x)$ is broken for any $\mu \in \mathcal{E}(\Phi)$.

In the same way we can construct a pair of weakly equivalent flows one of which has positive finite entropy and the other has infinite entropy. At last we emphasize that these examples can not be extended to differential flows. Because in these examples we have to make the ceiling function sharp. There is a still unsolved and interesting problem, whether the weak equivalence preserves the topological entropy in differentiable flows.

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