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Scattering of Lattice Solitons from A Mass Impurity

Nobuo YAJIMA

Research Institute for Applied Mechanics
Kyushu University, Fukuoka 812

A perturbation theory for the inverse scattering transforms applies to the Toda lattice system with a mass impurity. As an example, scattering of a soliton from an impurity is considered. When the mass of impurity is slightly different from that of host particles, the soliton nearly pass through the impurity without changing its amplitude.

§1 Introduction

Since Toda proposed a solvable model of one-dimensional lattice system, many works have been done on the lattice solitons.[1] Flaschka first applied the inverse scattering transforms to find the general method of solving the initial value problem for the infinite Toda lattice.[2]

Recently, the wave propagation in the anharmonic lattice system with impurities has been studied numerically and analytically, i.e., the reflection of a lattice-soliton from the mass-interface [3], effect of an impurity on propagation of a soliton [4], the excitation of localized mode due to the incidence of a soliton to the impurity [4], and so on. In this report, we study the propagation of solitons in the Toda lattice with a mass impurity by using the "inverse scattering transforms" method. Strictly speaking, this problem cannot be solved exactly by applying the inverse scattering transforms. This is because the time evolution of the bound state parameters just depends on the solution which is to obtain. However, the perturbation expansion for the inverse scattering transforms has existed.[5],[6] This perturbation theory applies to our problem.

The system concerned is subject to the Hamiltonian

$$H = \sum_n [P_n^2/m_n - \{\exp[-(Q_n-Q_{n-1})]-(Q_n-Q_{n-1})\}]. \qquad (1)$$

The P_n and Q_n are the momentum and the coordinate of the n-th particle. The mass of particle is taken to be unity except for n=0, i.e.,

$$m_n = (1 + \varepsilon\delta_{n,0})^{-1}, \tag{2}$$

where $\delta_{n,0}$ is Kronecker's symbol. Throughout the paper, we assume that the force strength is constant, that is, the interaction between the impurity and the host particles is common to that between host particles. The equations of motion is given by

$$\dot{Q}_n = \partial H/\partial P_n = P_n/m_n \ , \tag{3}$$

$$\dot{P}_n = -\partial H/\partial Q_n = -\{\exp[-(Q_{n+1}-Q_n)]-\exp[-(Q_n-Q_{n-1})]\}, \tag{4}$$

where the dot denotes time differentiation. In order to treat Eqs.(3) and (4) by the inverse scattering transforms, we introduce new variables instead of P_n's and Q_n's,[2)]

$$2a(n) = \exp[-(Q_n-Q_{n-1})/2], \ 2b(n) = -P_{n-1}. \tag{5}$$

It is readily seen that

$$\dot{a}(n) = [b(n+1)/m_n - b(n)/m_{n-1}]a(n), \tag{6}$$

$$\dot{b}(n) = 2[a(n)^2 - a(n-1)^2]. \tag{7}$$

In §2, Eqs.(6) and (7) are studied by the inverse scattering transforms. Owing to the existence of mass impurity, the time evolution of the scattering data is different from that of the pure Toda lattice system (see Appendix I). The effect of the mass impurity on soliton propagation is considered in §3 as the simplest example. This report is preliminary and the final one will be published with the detailed analyses.

§2 Inverse scattering transforms

We follow to Flaschka, defining a linear operator L,

$$[Lu](n) = a(n)u(n+1) + a(n-1)u(n-1) + b(n)u(n), \qquad (8)$$

and considering the eigenvalue problem,

$$[Lu](n) = \lambda u(n), \quad \lambda = (z+z^{-1})/2. \qquad (9)$$

We now assume that $a(n)=1/2$ and $b(n)=0$ for $n\to\pm\infty$.

If functions f and g are both solutions of Eq.(9), the Wronskian $W(f,g)_n$,

$$W(f,g)_n = a(n)\{f(n+1)g(n) - f(n)g(n+1)\}, \qquad (10)$$

is evidently independent of n.

Introduce the Jost functions $\phi(n,z)$ and $\psi(n,z)$, which satisfy Eq.(9) and possess the asymptotic forms such as

$$\phi(n,z) = z^{-n} \quad \text{for } n\to\infty, \qquad (11a)$$

$$\psi(n,z) = z^{n} \quad \text{for } n\to-\infty . \qquad (11b)$$

The Wronskian of these functions is

$$W(\psi(z),\psi(z^{-1}))_n = -W(\phi(z),\phi(z^{-1}))_n = (z-z^{-1})/2. \qquad (12)$$

The solution $\phi(n,z)$ is linearly independent of $\phi(n,z^{-1})$, and $\psi(n,z)$ is then expressed as a linear combination of them,

$$\psi(n,z) = \alpha(z)\phi(n,z^{-1}) + \beta(z)\phi(n,z), \tag{13}$$

where $\alpha(z)$ and $\beta(z)$ are the scattering data. Similarly we have

$$\phi(n,z) = \alpha(z)\psi(n,z^{-1}) - \beta(z^{-1})\psi(n,z). \tag{14}$$

For $|z|=1$, we have the relations

$$\beta(z^{-1}) = \beta^*(z), \quad \alpha(z^{-1}) = \alpha^*(z), \tag{15a}$$

$$|\alpha(z)|^2 = 1 + |\beta(z)|^2, \tag{15b}$$

where the asterisk denotes complex conjugation. For $|z|>1$, $\psi(n,z)z^{-n}$, $\phi(n,z)z^n$ and $\alpha(z)$ are analytic functions of z and the zeros of $\alpha(z)$ yield the discrete eigenvalue of Eq.(9).

According to the inverse scattering problem, a(n) and b(n) are obtained as

$$2a(n) = K_{n+1}/K_n, \quad 2b(n) = \kappa(n,n+1) - \kappa(n-1,n), \tag{16}$$

where $\kappa(n,m)$ and K_n satisfy the next Gel'fand-Levitan equation,

$$\kappa(n,m) + F(n+m) + \sum_{n'=n+1}^{\infty} \kappa(n,n')F(n'+m) = 0, \tag{17}$$

$$K_n^{-2} = 1 + F(2n) + \sum_{n'=n+1}^{\infty} (n,n')F(n'+n). \tag{18}$$

The kernel $F(\ell)$ is given by

$$F(\ell) = \frac{1}{2\pi i}\int_C z^{-\ell-1}\frac{\alpha(z)}{\beta(z)}dz, \tag{19}$$

where C denotes the integral contour closing anticlockwise and involving zeros of $\alpha(z)$ and the unit circle $|z|=1$. Performing the integration yields

$$F(\ell) = \sum_{r=1}^{N} C_r^2 z_r^{-\ell} + \frac{1}{2\pi i} \oint_{|z|=1} \frac{\beta(z)}{\alpha(z)} z^{-\ell-1} dz, \tag{20}$$

$$C_r^2 = \beta_r/(z_r \alpha_r'), \qquad \beta_r = \beta(z_r), \qquad \alpha_r' = (d\alpha/dz)_{z=z_r}, \tag{21}$$

where z_r's $(r=1,\cdots,N)$ are the zeros of $\alpha(z)$, $\alpha(z_r)=0$. We have also the following relations among $\phi(n,z)$, $\kappa(n,m)$ and K_n,

$$\phi(n,z) = K_n (z^{-n} + \sum_{m=n+1}^{\infty} \kappa(n,m) z^{-m}), \tag{22}$$

$$\kappa(n,m) = -\frac{1}{2\pi i K_n} \int_C \frac{\beta(z)}{\alpha(z)} \phi(n,z) z^{-(m+1)} dz, \tag{23}$$

$$K_n - K_n^{-1} = -\frac{1}{2\pi i} \int_C \frac{\beta(z)}{\alpha(z)} \phi(n,z) z^{-(n+1)} dz. \tag{24}$$

The time evolution of $\alpha(z)$, $\beta(z)$, z_r and C_r are given in Appendix I.

§3 Example

In order to obtain the solution a(n) and b(n) at an arbitrary time by solving the Gel'fand-Levitan equation, we must know the time evolution of scattering data, which can be easily obtained only for the case $\varepsilon=0$. In general, the scattering data evolves complicately, depending on the Jost functions $\phi(n,z)$ and $\psi(n,z)$ at an each instant. These Jost functions are determined by a(n,t) and b(n,t). Thus the method of solution falls into a circle.

Here we use the approximation that the eigenfunctions on the righthand sides of Eqs.(I-4)~(I-9) are replaced by those for $\varepsilon=0$. The simplest case is that of a one-soliton coming from far left to the mass impurity.

The eigenfunctions and the scattering data of one-soliton state are summarized in Appendix II. The time-evolutions of the bound state parameters are approximated by substituting the results of Appendix II into (I-6) and (I-8),

$$\dot{z}_1 = \frac{\varepsilon}{z_1-z_1^{-1}}[z_1+z_1^{-1}-\{\frac{A(1)}{A(0)}+\frac{A(0)}{A(1)}\}](A^2(0)-A^2(1)), \tag{25}$$

$$\dot{C}_1/C_1 = -(z_1-z_1^{-1})/2$$
$$+\frac{\varepsilon}{2(z_1-z_1^{-1})}\frac{1}{C_1^2 z_1}[(z_1+z_1^{-1})-(\frac{A(0)}{A(1)}+\frac{A(1)}{A(0)})]$$
$$\times[(z_1^2+C_1^2+1)A^2(0)-(z_1^4+z_1^2+3C_1^2)A^2(1)]. \tag{26}$$

If ε is small, we can estimate the change of z_1 and C_1 by the perturbation theory: Write

$$z_1 = z_1^{(0)}+\varepsilon z_1^{(1)}+\cdots,\quad C_1 = C_1^{(0)}+\varepsilon C_1^{(1)}+\cdots. \tag{27}$$

In the lowest order of ε, we find

$$\dot{z}_1^{(0)} = 0,\quad \dot{C}_1^{(0)} = -(z_1-z_1^{-1})/2. \tag{28}$$

Using these results, we proceed to the next order, to find $z_1^{(1)}$,

$$z_1^{(1)} = \frac{1}{z_1^{(0)} - z_1^{(0)^{-1}}} \int_{-\infty}^{\infty} dt [z_1^{(0)} + z_1^{(0)^{-1}} - \{\frac{\overset{(0)}{A}(1)}{\overset{(0)}{A}(0)} + \frac{\overset{(0)}{A}(0)}{\overset{(0)}{A}(1)}\}] (\overset{(0)2}{A}(0) - \overset{(0)2}{A}(1)),$$

where

$$\overset{(0)}{A}(n) = - \frac{(z_1^{(0)^2} - 1)^{-1/2}}{D^{(0)} z_1^{(0)n} + (D^{(0)} z_1^{(0)n})^{-1}} .$$

In view of $D^{(0)} \propto C_1^{(0)^{-1}} \propto \exp[(z_1^{(0)} - z_1^{(0)^{-1}})t/2]$ and $z_1^{(0)} =$ const., the integral is easily carried out. We finally obtain

$$z_1^{(1)} = 0 . \qquad (29)$$

That is to say, the change of the soliton amplitude is at most of the order ε^2 and the soliton nearly pass through the impurity. This tendency is seen in the numerical experiments for the incidence of small amplitude soliton on the impurity, in which the perturbation theory can be applied in good approximation. For the case of large ε, the naive perturbation method cannot be applied and Eqs.(25) and (26) must be carefully solved.

Appendix I Time dependence of $\alpha(z)$ and $\beta(z)$

Solving the equation obtained by differentiating Eq.(9) with respect to time,

$$[(L-\lambda)\dot{\psi}(z)](n) = -[\dot{L}\psi(z)](n),$$

we have

$$\dot{\psi}(n,z) = -\frac{2}{z-z^{-1}}\{\phi(n,z^{-1})\sum_{j=-\infty}^{n}\phi(j,z)[\dot{L}\psi(z)](j)$$

$$-\phi(n,z)\sum_{j=-\infty}^{n}\phi(j,z^{-1})[\dot{L}\psi(z)](j)\}, \quad \text{(I-1)}$$

where the boundary condition $\dot{\psi}(n,z)=0$ at $n=-\infty$ is imposed on account of the asymptotic form (11). We put $n\to\infty$ in (I-1) and use Eq.(13), to get

$$\dot{\alpha}(z) = -\frac{2}{z-z^{-1}}\sum_{n=-\infty}^{\infty}\phi(n,z)[\dot{L}\psi(z)](n), \quad \text{(I-2)}$$

$$\dot{\beta}(z) = \frac{2}{z-z^{-1}}\sum_{n=-\infty}^{\infty}\phi(n,z^{-1})[\dot{L}\psi(z)](n). \quad \text{(I-3)}$$

Substitution of Eqs.(6), (7) and (9), together with Eq.(2), yields

$$\dot{\alpha}(z) = -\frac{2\varepsilon}{z-z^{-1}}b(1)[a(0)\{\phi(0,z)\psi(1,z)+\phi(1,z)\psi(0,z)\}$$

$$-a(1)\{\phi(1,z)\psi(2,z)+\phi(2,z)\psi(1,z)\}] \quad \text{(I-4)}$$

$$\dot{\beta}(z) = -(z-z^{-1})\beta + \frac{2\varepsilon}{z-z^{-1}}b(1)[a(0)\{\phi(0,z^{-1})\psi(1,z)+\phi(1,z^{-1})\psi(0,z)\}$$

$$-a(1)\{\phi(1,z^{-1})\psi(2,z)+\phi(2,z^{-1})\psi(1,z)\}]. \quad \text{(I-5)}$$

Let z_r be a zero of (z), that is, $(z_r)=0$, and differentiate with respect to time. We then get

$$\alpha_r' \dot{z}_r = - [\partial\alpha(z)/\partial t]_{z=z_r} ,$$

where α_r' is defined in Eq.(21). The time evolution of the discrete eigenvalue z_r is thus obtained

$$\dot{z}_r = \frac{4\varepsilon z_r}{z_r - z_r^{-1}} C_r^2 b(1)[a(0)\phi_r(0) - a(1)\phi_r(2)]\phi_r(1), \quad \text{(I-6)}$$

where $\phi_r(n)=\phi(n,z_r)$ and $\psi_r(n)=\psi(n,z_r)=\beta_r\phi_r(n)$ are used and C_r, β_r are defined by Eq.(21).

Substituting (I-5) and (I-6) into the equation,

$$\dot{\beta}_r = d\beta(z_r)/dt = \beta_r' \dot{z}_r + [\partial\beta(z)/\partial t]_{z=z_r},$$

where $\beta_r' = [d\beta(z)/dz]_{z=z_r}$, and using the relation

$$\alpha_r'\phi(n,z_r^{-1}) = (d/dz)[\psi(n,z)-\beta(z)\phi(n,z)]_{z=z_r},$$

we obtain

$$\dot{\beta}_r = - (z_r - z_r^{-1})\beta_r$$

$$+ \frac{2\varepsilon}{z_r - z_r^{-1}} \frac{b(1)}{\alpha_r'} \frac{d}{dz}[a(0)\{\psi(0,z)\psi(1,z)-\beta_r^2\phi(0,z)\phi(1,z)\}$$

$$- a(1)\{\psi(1,z)\psi(2,z)-\beta_r^2 \phi(1,z)\phi(2,z)\}]_{z=z_r}. \quad \text{(I-7)}$$

It follows directly from (I-4), (I-6) and (I-7) that

$$\dot{C}_r/C_r = \frac{d}{dt} \ln[\beta_r/z_r\alpha_r']^{1/2}$$

$$= (1/2)[\dot{\beta}_r/\beta_r - \dot{z}_r/z_r - (\alpha''_r\dot{z}_r+[d\dot{\alpha}(z)/dz]_{z=z_r})/\alpha_r]$$

$$= -\ (z_r-z_r^{-1})/2 - ((z_r-z_r^{-1})^{-1} + \alpha''_r/2\alpha'_r)\dot{z}_r$$

$$+ \varepsilon b(1)/((z_r-z_r^{-1})\alpha'_r\beta_r)\frac{d}{dz}[V(0,z)-V(1,z)]_{z=z_r}, \qquad (I-8)$$

$$V(n,z) = a(n)[\psi(n,z)\psi(n+1,z) - \beta_r^2\phi(n,z)\phi(n+1,z)$$

$$+ \beta_r\{\phi(n,z)\psi(n+1,z)+\phi(n+1,z)\psi(n,z)\}]. \qquad (I-9)$$

Here we note that for the case $\varepsilon=0$ the time-evolutions of the bound state parameter reduce to that obtained by Flaschka except for the difference of the sign. This difference comes from the difference of asymptotic behaviors of the Jost functions.

Appendix II Eigenstate for a one-soliton state

We consider the case that $\beta(z)=0$ for $|z|=1$ and $\alpha(z)$ has only one zero, z_1, i.e., $\alpha(z_1)=0$, $|z_1|>1$. This case corresponds to the one soliton state unless the mass impurity exists. In this case, we have

$$F(\ell) = C_1^2 z_1^{-\ell}, \qquad (II-1)$$

$$C_1^2 = \frac{\beta_1}{z_1\alpha_1'}, = [\sum_{n=-\infty}^{\infty}\phi_1^2(n)]^{-1}.^{2)} \qquad (II-2)$$

The solution of the Gel'fand Levitan equation is given by

$$\kappa(n,m) = C_1A(n)z_1^{-m} \qquad m\geq n+1, \qquad (II-3)$$

$$K_n^{-2} = z_1 A(n)/A(n-1), \qquad \text{(II-4)}$$

$$A(n) = -(z_1^2-1)^{1/2}/[Dz_1^n + (Dz_1^n)^{-1}], \qquad \text{(II-5)}$$

$$D = (z_1^2-1)^{1/2}/C_1. \qquad \text{(II-6)}$$

We substitute (II-3) and (II-4) into Eq.(16) and then obtain

$$2a(n) = A(n)/(A(n+1)A(n-1))^{1/2}, \qquad \text{(II-7)}$$

$$2b(n) = [A(n-1)/A(n-2) - A(n)/A(n-1)]. \qquad \text{(II-8)}$$

Multiplying the eigenvalue equation (9) for $\psi(n,z)$ by $z^{-n}/\alpha(z)$ and taking the limit $|z|\to\infty$, we obtain

$$\lim_{|z|\to\infty} z^{-n}\psi(n,z)/\alpha(z) = 2a(n) \lim_{|z|\to\infty} z^{-(n+1)}\psi(n+1,z)/\alpha(z)$$

$$= \prod_{m=n}^{\infty} 2a(m)$$

From the asymptotic behavior of $\psi(n,z)$, we have

$$\lim_{|z|\to\infty} \alpha^{-1}(z) = \lim_{\substack{|z|\to\infty \\ n\to\infty}} z^{-n}\psi(n,z)/\alpha(z) = \prod_{m=-\infty}^{\infty} 2a(m) \qquad \text{(II-9)}$$

Substituting (II-7) into (II-9) gives

$$\lim_{|z|\to\infty} \alpha(z) = z_1^{-1} \qquad \text{(II-10)}$$

From the assumption that $\alpha(t)$ has a zero at $z=z_1$ and is analytic for $|z|>1$ and $|\alpha(z)|=1$ for $|z|=1$, we have

$$\alpha(z) = \frac{z-z_1}{zz_1-1}, \qquad \text{(II-11)}$$

$$\alpha_1' = (z_1^2-1)^{-1}, \qquad \alpha_1'' = -2z_1(z_1{}^2-1)^{-2}. \tag{II-12}$$

We use Eqs.(22), (II-3) and (II-4), to get

$$\phi_1(n) = \phi(n,z_1) = (C_1 z_1^{1/2})^{-1}[A(n-1)A(n)]^{1/2}. \tag{II-13}$$

From this, $\psi_1(n)$ is given by the relation $\psi_1(n)=\beta_1\phi_1(n)$; and $\beta_1=z_1\alpha_1'C_1^2 = C_1{}^2/(z_1-z_1^{-1})$,

$$\psi_1(n) = (C_1{}^2/(z_1-z_1^{-1}))\phi_1(n). \tag{II-14}$$

We obtain from Eqs.(22) and (II-3)

$$\phi(n,z) = K_n z^{-n}[\ 1 + C_1 A(n) z_1^{-n}/(zz_1-1)]. \tag{II-15}$$

Futher, we can obtain from the similar calculation

$$\psi(n,z) = (z_1 K_n)^{-1} z^n[1+(z_1-z_1^{-1})C_1^{-1}A(n-1)z_1^n/(zz_1-1)]. \tag{II-16}$$

By using (II-11) , (II -15) and (II-16) , we can show

$$\psi(n,z) = \alpha(z)\phi(n,z^{-1}). \tag{II-17}$$

Further, it can be directly shown from (II-14) and (II-15) that

$$\beta(z) = (2/(z-z^{-1}))W(\phi(z^{-1}),\psi(z))_n$$

$$= 0 \qquad \text{for} \quad |z|<|z_1|. \tag{II-18}$$

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