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Surjective morphisms of holomorphic vector bundles

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Let E and Q be holomorphic hermitian vector bundles over the complex analytic manifold X, and let g be

$$E \longrightarrow Q \longrightarrow 0,$$

a surjective morphism of vector bundles over X. We are interested in the fundamental and classical problem to give nice, sufficient conditions about X and E such that the induced morphism

$$g_* : H^0(X, E) \longrightarrow H^0(X, Q),$$

(over the global sections) will be also surjective. This is obviously true if, for instance, X is a Stein manifold. In this talk, we shall obtain such a result, using very weak positivity assumptions on X and E.

We suppose X kählerian and weakly pseudoconvex (i.e. there exists a plurisubharmonic exhaustive function of class C^2 over X). Let n be the complex dimension of X. K will denote the canonical line bundle of X. det Q is the line-bundle qQ (with q = rang Q). Let p be the rang of E. We have the following result:

Theorem. If E is semi-positive in the sense of Nakano (only in the sense of Griffiths if $p - q = 1$), then

$$g_* : H^0(X, E \otimes K \otimes (\det Q)^{\ell}) \longrightarrow H^0(X, Q \otimes K \otimes (\det Q)^{\ell})$$

is surjective for all integers $\ell > \inf(n, p - q)$.

The tangent bundle of the projective space P_n, the vector bundle $L \otimes \mathbb{C}^p$ (where L is a semi-positive line bundle) are examples of semi-positive vector bundles in the sense of Nakano.

The following result will be applied to a wide class of vector bundle.

Corollary. If E is the quotient bundle of some vector bundle F of rang N, which is semi-positive in the sense of Nakano, then

$$g_* : H^0(X, E \otimes K \otimes (\det Q)^{\ell}) \longrightarrow H^0(X, Q \otimes K \otimes (\det Q)^{\ell})$$

is surjective for all $\ell > \inf(n, N-q)$.

The assumption over E is realized for instance if E is generated by its global sections. For example, the tangent bundle of a homogeneous manifold is generated by its global sections. The corollary is a consequence of the theorem, putting up the section of Q directly in section of F.

Let S be the kernel of g, such that we have the exact sequence:

$$0 \longrightarrow S \longrightarrow E \longrightarrow Q \longrightarrow 0.$$

Considering the exact cohomology sequence, the surjectivity of g_* is equivalent to the vanishing of the image of the canonical morphism $\delta : H^1(X, Q) \longrightarrow H^1(X, S)$. It is a remarkable feature that we shall obtain the vanishing of the image of δ, but not necessarily the vanishing of all the group $H^1(X, S)$. Therefore our result may be thought as a partial vanishing theorem. Moreover it does not involve a strict positivity assumption, as it is usual in this kind of result.

In the proof we essentially use the fact that the curvature of the bundles S, E, Q are closely connected with the obstruction of the lifting of sections of Q in sections of E. Using curvature estimates, the Kodaira-Nakano inequality and Hörmander's L^2-methods, we solve some specific $\bar{\partial}$-equation with values in S, associated to the morphism δ. Moreover, we obtain some very precise L^2 estimates for the lift of sections of Q, involving the natural parameters of g. These estimates allow to obtain some more general results, concerning morphisms g, which are not everywhere surjective. Even locally these last results are not trivial. We do not reproduce them here for the simplicity. Complete results and proofs can be founded in [3]. Moreover, some extensions of the L^2 Hörmander estimates to non complete kähler and weakly pseudoconvex manifold are proved in [2].

The results of the papers [2] and [3] are in fact extensions of an older paper [1], whose motivations were completely different.

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[1] Application des techniques L^2 à la théorie des indéaux d'une algèbre de fonctions holomorphes avec poids, Annales Scient. de l'École Normale Superieure, 5, 1972, 545-579.

[2] Morphismes surjèctifs et fibrés lineàires semi-positifs, Seminaire P. Lelong-H. Skoda (Analyse) 17è anneé, 1976-77. Lecture Notes (à paraître).

[3] Morphismes surjectifs de fibrés vectoriels semi-positifs, préprint, Université Pierre et Marie Curie (Paris VI), 1978.