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Non-existence of bounded functions on the universal covering of $\{\mathbf{P}^n -$ n+2 hyperplanes in general position$\}$ $(n\geq 2)$.

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Let X be the complement of n+2 hyperplanes in general position in $\mathbf{P}^n$, and let $\pi:\tilde{X}\rightarrow X$ be the universal covering of X. Then we have

THEOREM 1. If $n\geq 2$, there are no bounded, holomorphic and non-constant functions on $\tilde{X}$. ([3])

We derive this theorem from the following theorem.

THEOREM 2. There are no bounded, holomorphic and non-constant functions on the "homological covering" of $\{\mathbf{P}^1 -$ 3 points$\}$. ([4])

1°) RELATION BETWEEN THE UNIVERSAL COVERING $\tilde{X}$ AND THE HOMOLOGICAL COVERING OF $\{\mathbf{P}^1 -$ 3 POINTS$\}$.

DEFINITION. The *homological covering* of a variety is the covering determined by the commutator group of the fundamental group of the variety.

Notation. We denote by R the homological covering of D= $\mathbf{P}^1-\{0,1,\infty\}$.

Let $z_1,\cdots, z_n$ be the inhomogeneous coordinates of $\mathbf{P}^n$. We may suppose that the n+2 hyperplanes in general position which we take away from $\mathbf{P}^n$ are $\{z_1=0\},\cdots,\{z_n=0\}$, $\{z_1+\cdots+z_n-1=0\}$ and the infinite hyperplane. Let $L(z_1,\cdots\overset{i}{\vee}\cdots,z_n)$ be the complex line consisting of the points whose j-th coordinate is equal to z_j, $j=1,\cdots,i-1,i+1,\cdots,n$.

Let us suppose that $z_1+\cdots+z_{i-1}+z_{i+1}+\cdots+z_n-1\neq 0$, then by a topological obsevation we have

PROPOSITION. Every connected component of the inverse image of $L(z_1,\ldots\overset{i}{\vee}\ldots,z_n)$ by π is holomorphically isomorphic to the homological covering $\mathbb{R}$ of D.

From this proposition and Theorem 2, the proof of Theorem 1 is immediate, because there are hense in $\tilde{X}$ sufficiently many analytic subsets admitting no bounded functions.

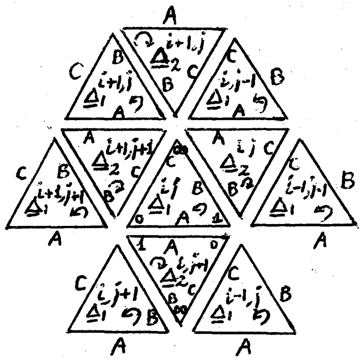
2°) TRIANGULATION OF R.

Let us pose $\Delta_1=\{z\in\mathbf{C}\mid \text{Im } z\geq 0,\ z\neq 0,1\}$ and $\Delta_2=\{z\in\mathbf{C}\mid \text{Im } z\leq 0,\ z\neq 0,1\}$. Let A be the segment $(0, 1)$, B be the segment $(1,\infty)$ and let C be the segment $(\infty, 0)$. We consider Δ_1 and Δ_2 as triangles with 3 sides A, B and C. The inverse image $\pi^{-1}(\Delta_1)$ of Δ_1 (resp. $\pi^{-1}(\Delta_2)$ of Δ_2) by the projection $\pi:R\longrightarrow D$ consists of an infinite number of triangles which are holomorphically isomorphic to Δ_1 (resp.Δ_2). We number the triangles of $\pi^{-1}(\Delta_1)$ and those of $\pi^{-1}(\Delta_2)$ as follows: Let p be a point in Δ_1. Let α (resp.β) be a closed curve of origin p and of end point p winding the point 0 (resp. the point 1) in the positive sense. Then every element of $\pi_1(D, p)/[\pi_1(D, p), \pi_1(D, p)]$ is represented uniquely by a curve of the form $\alpha^i\beta^j$ $(i,j\in\mathbf{Z})$. This element determines a point $p^{i,j}$ situated over p, so we denote by $\Delta_1^{i,j}$ the triangle of $\pi^{-1}(\Delta_1)$ which contains $p^{i,j}$. We further denote by $\Delta_2^{i,j}$ the triangle of $\pi^{-1}(\Delta_2)$ whose side B is identical with the side B of $\Delta_1^{i,j}$.

3°) TOPOLOGY OF R.

We prepare a countable number of triangles $\underline{\Delta}_1^{i,j}$ and $\underline{\Delta}_2^{i,j}$ $(i,j \in \mathbf{Z})$. We place them as Fig. 1.

Fig. 1.



We consider each pair of sides which are situated face to face in Fig. 1. We identify them oppositely so that the given orientations coinside. Let us denote by R' the surface constructed in this way. Then we can prove that the triangles $\Delta_1^{i,j}$ and $\Delta_2^{i,j}$ are patched in R exactly in the same way as the triangles $\underline{\Delta}_1^{i,j}$ and $\underline{\Delta}_2^{i,j}$ in R'. Hense

PROPOSITION. R is homeomorphic to R'.

4°) CRITERION OF A. PFLUGER.

To show Theorem 2 we use a criterion of A. Pfluger which gives a sufficient condition for a Riemann surface to belong to the class O_{AB}, (i.e. the class of surfaces with no bounded holomorphic functions.

Let W be an open Riemann surface and let $\{A_n^k\}$, $k=1,2,\cdots$, $k(n)<\infty$, $n=1,2,\cdots$, be a family of doubly connected domains in W satisfying the following conditions:

(1) A_n^k is bounded by two closed curves α_n^k and α'^k_n,

(2) $A_n^k \cap A_{n'}^{k'} = \phi$ if $(k,n) \neq (k',n')$,

(3) the complement of $\bigcup_{k=1}^{k(n)} A_n^k$ in W has a unique compact connected component B_n,

(4) B_n is bounded by k(n) curves $\{\alpha_n^k\}_{k=1}^{k(n)}$ and contains all $A_{n'}^{k'}$ such that $n' < n$.

Let us denote by μ_n^k the harmonic module of A_n^k, where the *harmonic module* of a domain conformally equivalent to $\{z \in \mathbf{C} \mid r < |z| < r'\}$ is by definition $\log r'/r$. We pose

$$\mu_n = \min_k \mu_n^k, \text{ and } K(N) = \max_{n \leq N} k(n).$$

CRITERION (A. Pfluger). ([1], [2]) If

$$\varlimsup_{N \to \infty} \left\{ \sum_{n=1}^{N} \mu_n - \frac{1}{2} \log K(N) \right\} = \infty,$$

then $W \in O_{AB}$.

5°) CONSTRUCTION OF POLYGONES $\{P_n\}$, AND ITS BOUNDARY.

We construct on our surface R a family of polygones $\{P_n\}$ such that

(i) each P_n consists of a finite number of triangles in 2°,

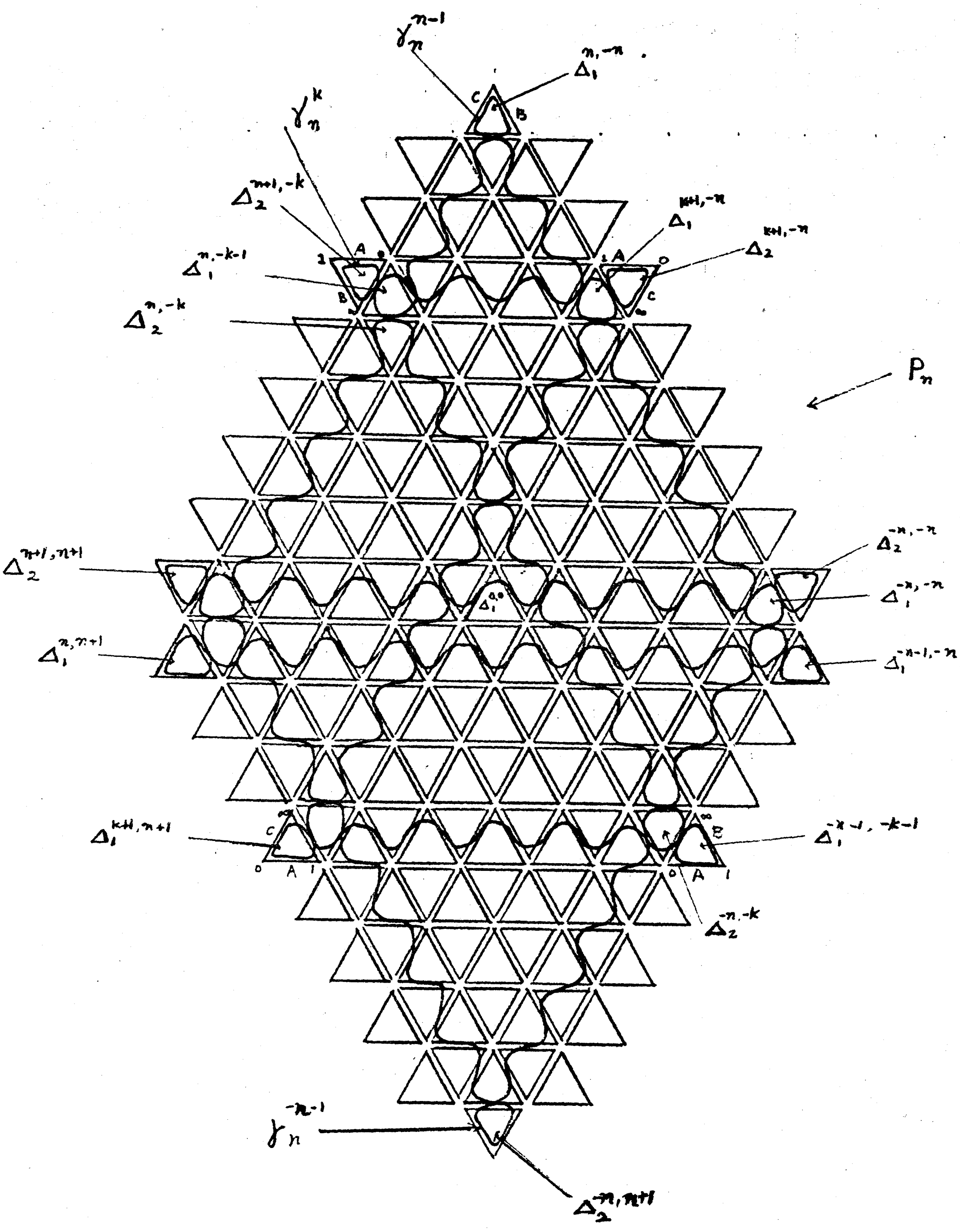
(ii) $P_n \subset P_{n+1}$,

(iii) $\bigcup_{n=1}^{\infty} P_n = R$.

Actually P_n is constructed as Fig.2.

We find that the boundary of P_n consists of 2n+1 connected

Figure 2.



γ_n^{n-1}
$\Delta_1^{n,-n}$
γ_n^k
$\Delta_2^{n+1,-k}$
$\Delta_1^{k+1,-n}$
$\Delta_2^{k+1,-n}$
$\Delta_1^{n,-k-1}$
$\Delta_2^{n,-k}$
P_n
$\Delta_2^{n+1,n+1}$
$\Delta_1^{n,n+1}$
$\Delta_1^{0,0}$
$\Delta_2^{-n,-n}$
$\Delta_1^{-n,-n}$
$\Delta_1^{-n-1,-n}$
$\Delta_1^{k+1,n+1}$
$\Delta_2^{-n,-k}$
γ_n^{-n-1}
$\Delta_2^{-n,n+1}$

components. For each connected component of the boundary of P_n, we describe a closed curve γ_n^k $(k=-n-1,\cdots,n-1)$ which runs sufficiently near to it.
$\{\gamma_n^k\}$ are given in Fig. 2.

6°) CONSTRUCTION OF DOUBLY CONNECTED DOMAINS $\{A_n^k\}$.

Let r, r' be two real numbers such that $r < r'$. Let us denote by $\Delta_\nu^{i,j}(0,r,r')$ the part of $\Delta_\nu^{i,j}$ $(\nu=1,2)$ lying over $\{z \in \mathbf{C} \mid r < |z| < r'\}$. Let $\Delta_\nu^{i,j}(1,r,r')$ be the part of $\Delta_\nu^{i,j}$ lying over $\{z \in \mathbf{C} \mid r < |z-1| < r'\}$. We pose $\Delta_\nu^{i,j}(\infty,r',r)=\Delta_\nu^{i,j}(0,r,r')$. Let $\Delta_\nu^{i,j}(0,r;1,r';\infty,r'')$ be the complement in $\Delta_\nu^{i,j}$ of the part lying over the set $\{|z|<r\} \cup \{|z-1|<r'\} \cup \{|z|>r''\}$.

We construct A_n^k as the union of the following sets. We take the set of the form $\Delta_\nu^{i,j}(0,r,r')$ (resp. $\Delta_\nu^{i,j}(1,r,r')$ and $\Delta_\nu^{i,j}(\infty,r',r)$) if $\Delta_\nu^{i,j}$ is situated in the interior of P_n and if γ_n^k passes near 0 (resp. 1 and ∞) in $\Delta_\nu^{i,j}$. And we take the set of the form $\Delta_\nu^{i,j}(0,r;1,r';\infty,r'')$ if γ_n^k passes the triangle $\Delta_\nu^{i,j}$ which is situated on the boundary of P_n. For example, in $\Delta_1^{i,-k-1}$ (resp. $\Delta_2^{i+1,-k}$) for $-n \leq i \leq n-1$, we take the part $\Delta_1^{i,-k-1}(0,r_{n+1},r_n)$ (resp. $\Delta_2^{i+1,-k}(0,r_{n+1},r_n)$), where $r_n=Ce^{-2\pi n}$ $(C>0)$. And in $\Delta_2^{n+1,-k}$ (resp. $\Delta_1^{n,-k-1}$), we take the part $\Delta_2^{n+1,-k}(0,r_{n+1};1,\varepsilon;\infty,1/r_{\frac{n-k+1}{2}})$ (resp. $\Delta_1^{n,-k-1}(0,r_{n+1};1,\varepsilon;\infty,1/r_{\frac{n-k-1}{2}})$) with $1 \gg \varepsilon > 0$. We put A_n^k the union of all these sets.

7°) HARMONIC MODULE OF A_n^k.

We divide A_n^k into some suitable parts, so that we have rectangles as their images by the map w=log z or w= log (z-1). We see that the width of each rectangle is 2π and the total length of all these rectangles is almost equal to $24\pi n$. Here $24\pi n$ is obtained as follows: A_n^k passes 12n+1 triangles among those which are in the interior of P_n. The intersection of A_n^k and each triangle of this kind is mapped by the above map to a rectangle of width 2π and of length π. So the total length of the intersection of A_n^k and all triangles in the interior of P_n is almost equal to $12\pi n$. On the other hand, we see that the total length of the intersection of A_n^k and all the triangles which are situated on the boundary of P_n is almost equal to $12\pi n$. $24\pi n$ is the sum of these quantities.

We can verify that these rectangles are patched together only slightly twisted and so A_n^k is almost like an annulus of width 2π and of length $24\pi n$ when we consider its harmonic module. Consequently the harmonic module μ_n^k of A_n^k is almost equal to $\log \frac{12n+2\pi}{12n}$, hense $\mu_n = \operatorname{Min}_k \mu_n^k$ is almost equal to $\log \frac{12n+2\pi}{12n}$.

Since $K(N) = \operatorname{Max}_{n \leq N} k(n) = 2N+1$, and $\pi/6 = 0.52\cdots > 1/2$, we obtain

$$\varlimsup_{N\to\infty} \left\{ \sum_{n=1}^{N} \mu_n - \frac{1}{2}\log K(N) \right\} \geqslant \varlimsup_{N\to\infty} \left\{ \sum_{n=1}^{N} \frac{\pi}{6n} - \frac{1}{2} \sum_{n=1}^{N} \left(\frac{\pi}{6n}\right)^2 - \frac{1}{2}\log(2N+1) \right\}$$

$$\geqslant \varlimsup_{N\to\infty} \left\{ \frac{\pi}{6}\log N - \frac{1}{2}\log(2N+1) - \frac{1}{2} \sum_{n=1}^{N} \left(\frac{\pi}{6n}\right)^2 \right\} = \infty .$$

Hense by virtue of the Pfluger's criterion, our Riemann surface R belongs to the class O_{AB} as required.

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