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Brody's Method in Value Distribution Theory

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In this thesis [B], R. Brody introduced a very clever method of reparametrizing holomorphic maps from the disc to a complex manifold which allowed the use of a normal families argument. The general principle which his method yields may be embodied in the following:

THEOREM 1. Let M, N be a compact complex manifolds (or more generally an analytic spaces), possibly with boundary, H, a differential metric on N. Then either

(1) There exists a non-constant holomorphic map $\mathbb{C} \xrightarrow{g} M$ such that $|dg(0)|_H = 1$ and $|dg(z)|_H \leq 1$ for all $z \in \mathbb{C}$

or

(2) Some neighborhood M_ε of M in N is hyperbolic; in fact, the infinitesimal Kobayashi metric is never zero on non zero tangent vectors.

Put in this form, the theorem has the following corollaries:

COROLLARY 1 ([B]). Any sufficiently small deformation of a hyperbolic compact complex manifold is hyperbolic.

Proof. If $\mathcal{M} \xrightarrow{\pi} \Delta$ is a deformation of $M = \pi^{-1}(0)$, take $N = \pi^{-1}(|z| < \frac{1}{2})$ and apply the theorem. As any subset of a hyperbolic space is hyperbolic, we conclude $M_t = \pi^{-1}(t)$ is hyperbolic for $|t|$ sufficiently small.

COROLLARY 2 ([B]). A compact complex manifold M is hyperbolic if and only if it admits no non-constant holomorphic map $\mathbb{C} \xrightarrow{g} M$ of order ≤ 2.

Proof. Take $N = M$, and note that $|dg(z)|_H \leq 1$ implies g has order ≤ 2.

Remark. If M is an algebraic variety containing an elliptic curve E, it is not hyperbolic and admits a non-constant holomorphic map of order 2, namely the porjection $\mathbb{C} \longrightarrow \mathbb{C}/\Lambda = E$. Thus order ≤ 2 is the best statement possible.

COROLLARY 3 ([G]). If M is an analytic subspace of a complex torus T and if M contains no elliptic curves, then M is hyperbolic.

Proof. Take $N = M$ and take H to be the Euclidean metric. If alternative (1) holds, lift g so that we have

$$\begin{array}{ccc} & & \mathbb{C}^n \\ & {\scriptstyle G} \nearrow & \\ \mathbb{C} & \xrightarrow{g} & \mathbb{C}^n/\Lambda = T \end{array} \qquad G = (G_1, \cdots, G_n)$$

Then $|G_1'|^2 + \cdots + |G_n'|^2 < 1$, so by Liouville's theorem G_1' = constant for all i, thus $G(z) = az + b$, where $a, b \in \mathbb{C}^n$, i.e. g is a translate of a one-parameter subgroup of T. So $\overline{g(\mathbb{C})}$ gives a non-trivial real subtorus contained in M, and hence a non-trivial complex subtorus and a fortiori an elliptic curve. This is forbidden by hypothesis, so alternative (1) is eliminated, leaving alternative (2).

Brody's method can be generalized to the non-compact case either by making use of Hurwitz's classical theorem on limits of nowhere zero functions, as done by Alan Howard, or by taking a complete metric with certain properties, as in [G2].

The general theorem one obtains is:

THEOREM 2. Let M be a compact complex manifold and D an analytic subset of M with normal crossings. Let $X_k = \{z \in M \mid \text{multiplicity}_D(z) = k\}$. Then either

(1) $M - D$ is complete hyperbolic and hyperbolically embedded in M

or

(2) For some $k \geq 0$, there exists a non-constant holomorphic map $\mathbb{C} \xrightarrow{g} X_k$ of order ≤ 2.

As corollaries we have

COROLLARY 1. Let $H_1, \cdots, H_{2n+1}$ be hyperplanes in general position in $\mathbb{P}_n$. Then $\mathbb{P}_n - (H_1 \cup \cdots \cup H_{2n+1})$ is complete hyperbolic and hyperbolically embedded in $\mathbb{P}_n$.

Proof. The irreducible components of X_k are copies of $\mathbb{P}_{n-k}$ - ($2n+1-k$ hyperplanes in general position in $\mathbb{P}_{n-k}$). Noting $2n+1-k \geq 2(n-k)+1$, alternative (2) is excluded by the Picard theorem for projective space minus $2n+1$ hyperplanes in general position.

COROLLARY 2. Let $\chi \xrightarrow{\pi} \Delta^{3g-3}$ be a local universal deformation over the unit polycylinder of dimension $3g - 3$ of a stable curve $\pi^{-1}(0)$ be complete algebraic curves of genus ≥ 2. Let $\chi_\varepsilon = \pi^{-1}(\Delta(1 - \varepsilon))$ and $\chi_\varepsilon^{\text{sing}}$ be the singular fibres in χ_ε. Then for any $\varepsilon > 0$, $\chi_\varepsilon - \chi_\varepsilon^{\text{sing}}$ is complete hyperbolic and hyperbolically embedded in χ_ε.

Proof. As all of the X_k project to subsets of Δ^{3g-3}, any holomorphic map $\mathbb{C} \xrightarrow{g} X_k$ goes to a fibre. But the fibres of

$X_k \xrightarrow{\pi} \pi(X_k)$ are unions of punctured curves $M - \{p_1, \cdots, p_k\}$ so that $\chi(M) + k > 0$. By the classical Picard's theorem, there are no non-constant holomorphic mpas $\mathbb{C} \xrightarrow{g} M - \{p_1, \cdots, p_k\}$.

There are other results along the lines Corollary which follow by these methods. For example, one can obtain a proof of Mordell's conjecture for families of curves (Grauert, Manin) in this way.

COROLLARY 3 ([G]). Let T be a complex torus, D an analytic hypersurface which contains no elliptic curve. Then $T - D$ is complete hyperbolic and hyperbolically embedded in T.

Proof. This is actually a corollary of the method of proof rather than the theorem itself. We obtain that the map $\mathbb{C} \xrightarrow{g} T - D$ of alternative (2) is distance-decreasing with respect to a complete metric, and this forces g to be constant. There can be no non-constant holomorphic maps $\mathbb{C} \xrightarrow{g} D$ by Corollary 3 of Theorem 1. So alternative (1) holds.

References

[B] R. Brody, Thesis (Harvard)

[G1] M. Green, "Holomorphic maps to complex tori", to appear, Am. J. Math.

[G2] M. Green, "The hyperbolicity of the complement of 2n+1 hyperplanes in general position in $\mathbb{P}_n$, and related result", Proc. A. M. S. 66(1977), 109-113.