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# ON REDUCIBLE PLANE CURVES

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## §1. Introduction.

Let $f:\mathbb{C}^2 \longrightarrow \mathbb{C}$ be a polynomial function such that $f(0)=0$ and f has an isolated critical point at the origin, and let L be the intersection of $V=f^{-1}(0)$ with a small sphere $S^3_\varepsilon$. If f is analytically irreducible, then L is an iterated torus knot whose type is determined by the Puisex characteristic pairs associated for f. If f is reducible and factored by irreducible components $f_i$ as $f=f_1\cdots\cdots f_r$, then L is a link of r components whose components are associated knots for $f_i$. Let $K_i$ be the intersection of $V_i=f_i^{-1}(0)$ with a small sphere $S^3_\varepsilon$ for $i=1,\cdots\cdots,r$. For the complement X of L (in $S^3$), $H_1(X;Z)$ is a free abelian multiplicative group G on the symbols $\{t_i\}_{i=1}^{r}$, where each $t_i$ is geometrically carried by a meridian circle with homological linking number $\delta_{ij}$ with the component $K_j$ of L. Let $\Lambda_r$ be the integral group ring ZG, and let $\tilde{X}$ be the universal abelian covering of X. Then $H_1(\tilde{X};Z)$ is a $\Lambda_r$-module. The Alexander polynomial of L is defined to be the characteristic polynomial of $H_1(X;Z)$ and it is topological invariant of L (up to units in $\Lambda_r$).

Lê [1] showed that algebraic knots having the same Alexander polynomial (mod. unit) are of the same knot type. In this paper we shall show the following; let L be an algebraic link of $r(\geq 2)$ components whose components have distinct tangent cones each other.

THEOREM A. Let L and L′ be as above. If L and L′ have the same Alexander polynomial (mod. units), then They are of the same link type.

§2. Algebraic links and Alexander polynomials.

First of all, we review algebraic knots and associated Puiseux expansions. We mainly refer to F. Pham [3].

Let $f:\mathbb{C}^2 \to \mathbb{C}$ be an analytically irreducible polynomial such that $f(0)=0$ and f has an isolated critical points at the origin. Then f can be written as

$$f(x,y)=\prod_{\zeta^m=1}(y-y(\zeta x))$$

in a small neiborhood of the origin, where m is the order of y and $\zeta$'s are all m-th roots of unity, and

$$y(\zeta x)=\sum_{n=1}^{+\infty} a_n \zeta^n x^{n/m} \qquad a_n \in \mathbb{C}.$$

Let $\mu=n_0/m$ be the exponent of the first term of $y(\zeta x)$ which has a non-zero coefficient. By $in(y-y(\zeta x))$, we denote the terms of $y-y(\zeta x)$ which have the smallest degree; that is,

$$in(y-y(\zeta x))=\begin{cases} y & \mu>1, \\ y-a_0 x & \mu=1, \\ -a_{n_0}\zeta^{n_0}x^{n_0/m} & \mu<1. \end{cases}$$

DEFINITION. The tangent cone $C(V,0)$ of V at the origin is the limit of the tangents at $v_j$, where $v_j$ are the points of V tending to the origin.

Since $in(f)=in(\prod_{\zeta^m=1}(y-y(\zeta x)))=\prod_{\zeta^m=1} in(y-y(\zeta x))$, then

$$in(f)=\begin{cases} y^m & \mu>1, \\ (y-a_{n_0}x)^m & \mu=1, \\ \eta(-a_{n_0})^m x^{n_0} & \mu<1, \text{ where } \eta=\prod_{\zeta^m=1}\zeta^{n_0}. \end{cases}$$

Then the tangent cone $C(V,0)$ is the line defined by $in(f)=0$.

By changing coordinates linearly, we take the x-axis as $C(V,0)$. Then, assuming that $C(V,0)$ is the x-axis and by taking $\zeta=1$, we have the Puiseux expansion of f

$$y=p(x)+a_1x^{n_1/m_1}+\sum_{j=1}^{\ell_1}a_{1j}x^{n_1+j/m_1}+a_2x^{n_2/m_1m_2}+\sum_{j=1}^{\ell_2}a_{2j}x^{n_2+j/m_1m_2}$$

$$+\cdots\cdots+a_qx^{n_q/m_1\cdots m_q}+\sum_{j=1}^{+\infty}a_{q,j}x^{n_q+j/m_1\cdots m_q},$$

where p(x) is a polynomial of x. We call the sequence of pairs of relatively prime positive numbers $\{(n_j,m_j)\}_{j=1}^{q}$ the Puiseux characteristic pairs of f.

Let K be the associated knot $V\cap S^3_\varepsilon$ for f. Then K is the iterated torus knot of type $\{(m_j,\lambda_j)\}_{j=1}^{q}$ inductively constructed as follows; let $K^0$ be the unknotted circle $C(V,0)\cap S^3_{\varepsilon'}$, where $\varepsilon'$ is the positive number such that $\varepsilon'<\varepsilon$ and $\varepsilon'$ is sufficiently close to $\varepsilon$. We call $K^0$ the primitive core of K. Let $K^1$ be the torus knot of type $(m_1,n_1)$ in a small tubular neiborhood of $K^0$, where the first coordinate $m_1$ and the second coordinate $n_1$ are the longitudinal winding number and the meridianal winding number respectively. This notation is opposite to that of Lê [1] and Sumners and Woods [4]. We suppose that the (q-1)-st iteration $K^{q-1}$ has been constructed. Let T and $T_{q-1}$ be an unknotted torus and a small tubular neiborhood of $K^{q-1}$ respectively. Let $\varphi: T \longrightarrow T_{q-1}$ be the orientation preserving diffeomorphism from T to $T_{q-1}$ which carries the longitude to the longitude. Then K is defined to be the image $K^q = \varphi(k)$ of a torus knot k of type $(m_q, \lambda_q)$ in T, where

(2.1) $$\lambda_1 = n_1$$

$$\lambda_j = n_j - n_{j-1} m_j + \lambda_{j-1} m_{j-1} m_j \qquad j=2, \cdots, q.$$

Next, we consider the case that f is reducible. Let $f: \mathbb{C}^2 \rightarrow \mathbb{C}$ be a polynomial such that $f(0)=0$ and f has an isolated critical point at the origin. We suppose that f is factored by irreducible components $f_i$ as $f = f_1 \cdots f_r$ $(r \geq 2)$. Let the Puiseux characteristic pairs of $f_i$ be $\{(n_{ij}, m_{ij})\}_{j=1}^{q_i}$, and let $K_i$ be the associated iterated torus knot of type $\{(m_{ij}, \lambda_{ij})\}_{j=1}^{q_i}$ for $f_i$. In this paper, we consider the case that all tangent cones $(V, 0)$ $(V_i = f_i^{-1}(0))$ are distinct each other. For the general case, refer to Sumners and Woods [4] and Yamamoto [6]. We denote the associated link for f by L. Then L is the disjoint union of $K_1$, $\cdots, K_r$ constructed as follows; let $L^0$ be the link consisting of all $K_i^0 = C(V_i, 0) \cap S_\varepsilon^3$ $(i=1, \cdots, r)$. We note that $L^0$ has the same link type of the torus link of type $(r, r)$. Then L is the link obtained by iteration of type $\{(m_{ij}, \lambda_{ij})\}_{j=1}^{q_i}$ on each $K_i^0$ for i=

$1,\cdots,r$.

Let $\ell=k_1\cup\cdots\cup k_r$ be a link of $r$ components in $S^3$, and $\ell'$ be the link obtained from $\ell$ by iteration of type $(m,\lambda)$ on a component $k_r$, where $m>1$. Let $\Delta(\ell;t_1,\cdots,t_r)$ and $\Delta(\ell';t_1,\cdots,t_r)$ be the Alexander polynomials of $\ell$ and $\ell'$ respectively. Sumners and Woods [4] proved the following useful theorem.

THEOREM (Sumners and Woods [4] 5.1.) Let $\ell$ and $\ell'$ be as above. Then we have

(2.2) $\Delta(\ell';t_1,\cdots,t_r)=\Delta(\ell;t_1,\cdots,t_r^{m})Q(t_r,y:m,\lambda)$,

where $y=\sum_{i=1}^{r-1} t_i^{\langle k_i,k_r\rangle}$, $\langle k_i,k_r\rangle$ is the linking number of $k_i$ and $k_r$ and $Q(t,s;m,\lambda)$ is the Alexanderpolynomial of two components formed by the torus knot $k$ of type $(m,\lambda)$ and the unknotted meridian curve on the bpundary torus containing $k$,

(2.3) $Q(t,s;m,\lambda)=((t^{\lambda}s)^{m}-1)/(t^{\lambda}s-1)$.

Let $L$ be the algebraic link associated for a reducible polynomial $f=f_1\cdots f_r$ whose tangent cones are distinkt each other. Since $L^{o}$ has the same link type of the torus link of type $(r,r)$, we have

$$\Delta(L^{o};t_1,\cdots,t_r)=(t_1\cdots t_r-1)^{r-2}.$$

Then we can compute the Alexander polynomial $\Delta(L;t_1,\cdots,t_r)$ of $L$ by (2.2). Let $\nu_{i;j,k}$ and $y_{ij}$ be

$$\nu_{i;j,k}=\begin{cases} m_{i,j}\cdots m_{i,k} & 1\leq j\leq k\leq q_i \\ 1 & j>k,\end{cases}$$

and

$$y_{i,j}=\prod_{\substack{\ell=1\\ \ell\neq i}}^{r} t_\ell^{\nu_{\ell;1.q}\,\nu_{i;1,j-1}},$$

respectively for $i=1,\cdots,q$. Then we have

LEMMA 2.1. *The Alexander polynomial of* L *is*

(2.4) $$\Delta(L;t_1,\cdots,t_r)$$
$$=\left(\prod_{i=1}^{r} t_i^{\nu_{i;1,q_i}}-1\right)^{r-2}\prod_{i=1}^{r}\prod_{j=1}^{q_r} Q(t_i^{\nu_{i;j+1,q_i}},y_{ij};m_{ij},\lambda_{ij})$$

§3. Proof of THEOREM A.

Let $\Phi(t;m)$ be a polynomial of the form

(3.1) $$\Phi(t;m)=t^{m-1}+t^{m-2}+\cdots+t+1,$$

where m is a positive integer. We call a polynomial of the form (3.1) $\Phi$-polynomial. Then a plynomial $Q(t,s;m,\lambda)$ is a $\Phi$-polynomial

$$Q(t,s;m,\lambda)=\Phi(t^{\lambda}s;m).$$

Therefore the Alexander polynomial $\Delta(L;t_1,\cdots,t_r)$ of the link L can be written as

$$(3.2)\quad \Delta(L;t_1,\cdots,t_r)=(\prod_{i=1}^{r} t_i^{\xi_i}-1)^{r-2}\,\Phi(\prod_{i=1}^{r} t_i^{\xi_i};\ g)$$
$$\prod_{i=1}^{r}\prod_{j=1}^{q_i}\Phi(t_i^{\lambda_{ij}\nu_{i;j+1,q_i}}\ y_{ij};\ m_{ij}),$$

where $g$=g.c.m.$\{\nu_{i;1,q_i}\}_{i=1}^{r}$ and $\xi_i=\nu_{i;1,q_i}/g$ for $i=1,\cdots,r$.

Before proceeding the proof of THEOREM A, we preperate several lemmas. Let $\varphi_1=\Phi(t_1^{\omega_1}\cdots t_r^{\omega_r};m)$ and $\varphi_2=(t_1^{\omega'_1}\cdots t_r^{\omega'_r};m')$ for integers $m$ and $m'\geqq 2$. The following LEMMA 3.1 and LEMMA 3.2 are showed directly.

LEMMA 3.1. *The product $\varphi_1\cdot\varphi_2$ is a $\Phi$-polynomial if and only if $\omega_i\neq\omega'_i$, and $\omega_i m=\omega'_i$ if $\omega_i<\omega'_i$ or $\omega_i=\omega'_i m'$ if $\omega_i>\omega'_i$, for all* $i=1,\cdots,r$.

Let $\varphi_i=\varphi_{i,1}\cdots\varphi_{i,u_i}$ $(i=1,2)$ be a decomposition of $\varphi_i$ by $\Phi$-polynomials $\varphi_{i,j}$. We say that $\varphi_1$ and $\varphi_2$ are *relatively prime* if a product $\varphi_{1,j}\varphi_{2,j'}$ is not a $\Phi$-polynomial for any $j=1,\cdots,u_1$ and any $j'=1,\cdots,u_2$.

We suppose that $m$ is factored by prime integers as $m=a_1\cdots a_u$. Then we have

LEMMA 3.2. *A $\Phi$-polynomial $\Phi(t;m)$ is factored by (irreducible) $\Phi$-polynomials as*

$$\Phi(t;m)=\prod_{j=1}^{u}\Phi(t^{\alpha_j};a_j),$$

*where* $\alpha_1=1$ *and* $\alpha_j=a_1\cdots a_{j-1}$ *for* $j=2,\cdots,u+1$.

We note that an iterated torus knot K of type $\{(m_j,\lambda_j)\}_{j=1}^{q}$ is algebraic if and only if $n_{j-1}m_j<n_j$ for all $j=2,\cdots,q$, where $n_j$ are given by (2.1). Therefore, for all j and k such that $1\leq j<k\leq q$, we have

$$\lambda_k>\lambda_j m_j\cdots m_k. \tag{3.3}$$

To prove THEOREM A, it is sufficient to show that we can determine the knot types of components $K_i$ $(i=1,\cdots,r)$ of L from the Alexander polynomial $\Delta(L;t_1,\cdots,t_r)$ of L.

LEMMA 3.3. *Let* $\varphi_{ij}=(t_i^{\lambda_{i,j}\nu_{i;j+1,q_i}}y_{ij};m_{ij})$ *and* $\varphi_{ik}=(t_i^{\lambda_{i,k}\nu_{i;k+1,q_i}}y_{ik};m_{ik})$. *Then* $\varphi_{ij}$ *and* $\varphi_{ik}$ *are relatively prime for* $i=1,\cdots,r$ *and* $1\leq j<k\leq q_i$.

Proof. By (3.3), for all j and k such that $1=j\ k=q_i$,

$$\lambda_{ik}\nu_{i;k+1,q_i}>\lambda_{ij}\nu_{i;j+1,q_i}.$$

Then by LEMMA 3.1, $\varphi_{ij}$ and $\varphi_{ik}$ are relatively prime.

LEMMA 3.4. $\varphi_{ij}$ *and* $\varphi_{\ell k}$ *are relatively prime for* $i,\ell=1,\cdots,r$, $i\neq\ell$, $j=1,\cdots,q_i$ *and* $k=1,\cdots,q_\ell$.

Proof. We suppose that $m_{ij}$ and $m_{\ell k}$ are factored by prime integers as $m_{ij}=a_1\cdots a_u$ and $m_{\ell k}=b_1\cdots b_v$ respectively. Let $\alpha_p$ and $\beta_s$ be $\alpha_1=1$, $\alpha_p=a_1\cdots a_{p-1}$ $(2\leq p\leq u+1)$, $\beta_1=1$ and $\beta_s=b_1\cdots b_{s-1}$ $(2\leq s\leq v+1)$. We suppose that $\varphi_{ij}$ and $\varphi_{\ell k}$ are not relatively prime.

Then there are $p(1\leq p\leq u)$ and $s(1\leq s\leq v+1)$ such that

(3.4) $\lambda_{ij}\nu_{i;j+1,q_i}\alpha_p=\nu_{i;1,q_i}\nu_{\ell;1,k-1}\beta_s$

or there are $p(1\leq p\leq u+1)$ and $s(1\leq s\leq v)$ such that

(3.5) $\nu_{;1,q_\ell}\nu_{i;1,j-1}\alpha_p=\lambda_{\ell k}\nu_{\ell;k+1,q_\ell}\beta_s.$

Since $\alpha_p<m_{ij}$ for $1\leq p\leq u$ and $\beta_s<m_{\ell k}$ for $1\leq s\leq v$, $\bar{\alpha}_p=m_{ij}/\alpha_p>1$ and $\bar{\beta}_s=m_{\ell k}/\beta_s>1$. Then by (3.4),

$$\lambda_{ij}=\bar{\alpha}_p\nu_{\ell;1,k-1}\beta_s$$

or by (3.5),

$$\lambda_{\ell k}=\bar{\beta}_s\nu_{i;1,j-1}\alpha_p$$

These contradict that g.c.d.$(m_{ij},\lambda_{ij})=1$ or g.c.d.$(m_{\ell k},\lambda_{\ell k})=1$. Then by LEMMA 3.1, $\varphi_{ij}$ and $\varphi_{\ell k}$ are relatively prime.

<u>LEMMA</u> 3.5. <u>$\varphi_{ij}$ and $\Phi(\prod_{i=1}^{r} t_i^{\xi_i};g)$ are relatively prime for $i=1,\cdots,r$, and $j=1,\cdots,q_i$.</u>

Proof. By (3.3) and since $m_{i,1}<\lambda_{i,1}$, for any i and any j,

$$\nu_{i;1,q_i}<\lambda_{ij}\nu_{i;j+1,q_i}$$

Then by LEMMA 3.1, $\varphi_{ij}$ and $\Phi(\prod_{i=1}^{r} t_i^{\xi_i};g)$ are relatively prime.

By lemmas 3.3, 3.4 and 3.5, the Alexander polynomial $\Delta(L;t_1,\cdots,t_r)$ of L is uniquely representd by the form (3.2). Therefore we can uniquely determine the knot types of components $K_i$ of L from the exponents of $t_i$ in (3.2). This completes the proof.

§4. Reduced Alexander polynomials.

Let $\ell$ be a link of r components. Then the reduced Alexander polynomial $\Delta(\ell;t)$ is given by the equation $\Delta(\ell;t)=(t-1)\Delta(\ell;t,\cdots,t)$ (see Milnor [3]). The following example shows that algebraic links can not be classified by reduced Alexander polynomials.

EXAMPLE 4.1. Let $f(z_0,z_1)=z_0(z_0^5-z_1^6)$ and $f'(z_0,z_1)=(z_0^3-z_1^2)(z_0^3-z_1^{10})$. Then associated links L and L′ (for f and f′ respectively) have the same reduced Alexander polynomial $(L;t)=\Delta(L';t)=(t-1)(t^6+1)$. But L and L′ are not of the same link type.

Let $\Gamma$ and $\Gamma'$ be the Seifert matrices of L and L′ respectively. From Amida-diagrams of L and L′ (see [5]), we can compute $\Gamma$ and $\Gamma'$. Then by computations with a computer, we have that the signatures $\sigma(L)$ of L and $\sigma(L')$ of L′ are $\sigma(L)=19$ and $\sigma(L')=23$.

EXAMPLE 4.2. Let $g(z_0,\cdots,z_n)=z_0(z_0^5-z_1^6)+z_2^2+\cdots+z_n^2$ and $g'(z_0,\cdots,z_n)=(z_0^3-z_1^2)(z_0^3+z_1^{10})+z_2^2+\cdots+z_n^2$. Let K and K′ be associated knots for g and g′ respectively. Then Alexander polynomial $\Delta(K;t)$ and $\Delta(K';t)$ are also equal to $(t-1)(t^6+1)$. But K and K′ are not of the same knot type because signatures $\sigma(K)=19$ and $\sigma(K')=23$. Therefore algebraic knots $K^{2n-1}$ can not be classified by Alexander polynomials.

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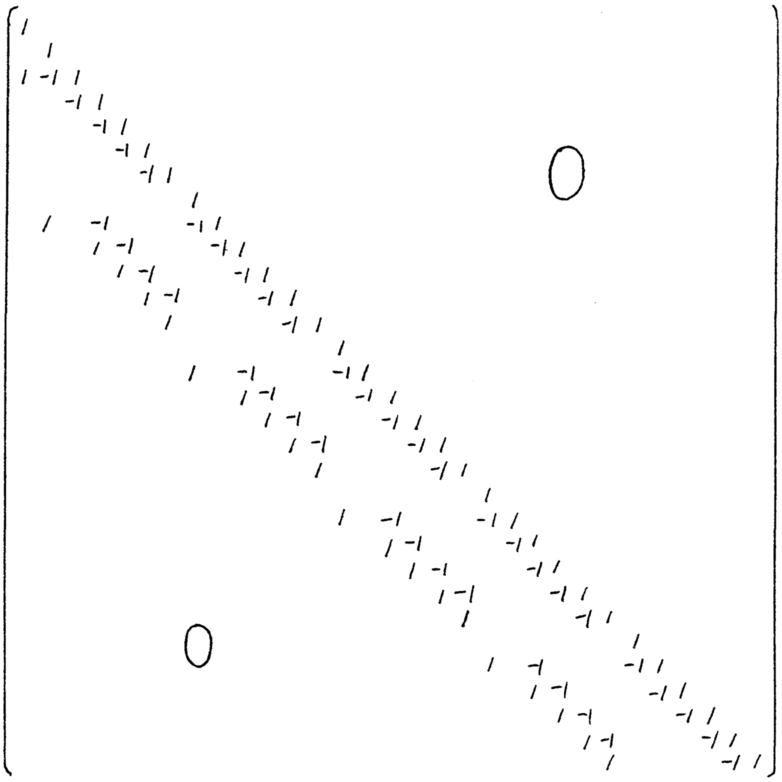
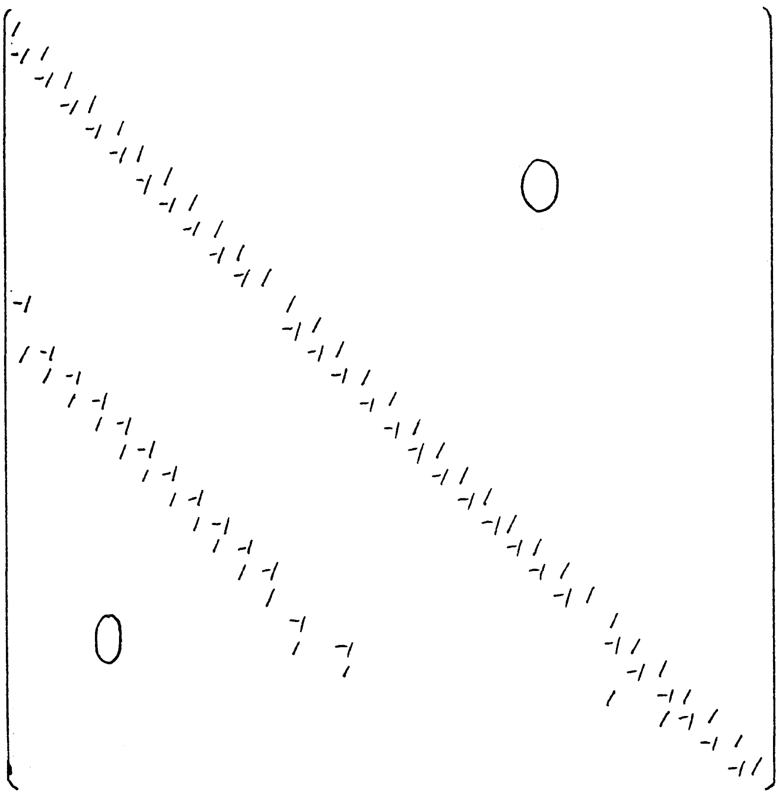
$\Gamma =$



$\Gamma' =$



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