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Towards a Spectral Theory on an Indefinite Inner Product Space

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Let $(H, \langle\cdot|\cdot\rangle)$ be an indefinite inner product space, that is a real or complex vectorspace H equipped with a bilinear respective sesquilinear, hermitian and non-degenerate form $\langle\cdot|\cdot\rangle$. A norm p on H is said to be selfpolar if

$$p(x) = \sup_{p(y)\le 1} |\langle x|y\rangle| \qquad \forall x \in H$$

Usually $(H, \langle\cdot|\cdot\rangle)$ allows many different and even non-equivalent selfpolar norms. However, when $\langle\cdot|\cdot\rangle$ is definite the inner square gives rise to a norm $\tau(x) = |\langle x|x\rangle|^{\frac{1}{2}}$ which is the unique selfpolar norm on H. We will henceforth assume that H allows a Hilbert space structure $(\cdot|\cdot)$ making $\langle\cdot|\cdot\rangle$ continuous.

Proposition 1 Let H^+ and H^- be subspaces of H such that

(i) $\langle\cdot|\cdot\rangle$ is positive (negative) definite on $H^+(H^-)$

(ii) $(H^+)^\perp = H^-$, $(H^-)^\perp = H^+$

(iii) $H^+ \oplus H^-$ is dense in H

(iv) H^+ and H^- are regular

There is a unique selfpolar quadratic norm p on H such that

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$$p(x) = \sqrt{\langle x^+ | x^- \rangle - \langle x^- | x^- \rangle}$$

for $x \in H^+ \oplus H^-$, $x = x^+ + x^-$, $x^\pm \in H^\pm$

The norm p introduced in Proposition 1 is said to be a quasi-decomposition norm and the pair (H^+, H^-) is referred to as the quasi-decomposition of H corresponding to p. Associated with p we introduce a closed involution J with domain $D(J) = H^+ \oplus H^-$ defined by

$$J(x^+ + x^-) = x^+ - x^- \quad , \quad x^\pm \in H^\pm$$

We refer to [1] for a detailed discussion of quasi-decomposition norms and their characterisation in the set of quadratic selfpolar norms.

<u>Theorem 2</u> Let p_1 and p_2 be quasi-decomposition norms with corresponding quasi-decompositions (H_1^+, H_1^-) and (H_2^+, H_2^-). If $H_1^+ \oplus H_1^- = H_2^+ \oplus H_2^-$, then p_1 and p_2 are equivalent.

Let A be a linear operator on $(H, \langle\cdot|\cdot\rangle)$ with dense domain D(A). We define

$$D(A)^\# = \{y \in H | \exists y_1 \in H : \langle Ax|y\rangle = \langle x|y_1\rangle \ \forall \ x \in D(A)\}$$

and put $A^\# y = y_1$ for $y \in D(A^\#)$. Because $\langle\cdot|\cdot\rangle$ is non-degenerate and D(A) is dense, this definition makes sense and gives a linear operator $A^\#$ called the adjoint of A with respect to $\langle\cdot|\cdot\rangle$.

Remarks 3

1) $A^{\#}$ can be unbounded (even non-densely defined) for a bounded A

2) A projection P, that is $P^2 = P$, $P^{\#} = P$ can be unbounded

3) $A = A^{\#} \in L(H)$ can be nilpotent

Case 1) and 2) can happen even when $\langle\cdot|\cdot\rangle$ is definite (non-complete) while 3) is a phenomenon occurring only in manifestly indefinite inner product spaces.

Definition 4 An operator A is said to be reduced by a quasi-decomposition norm p if

$$\text{(i)} \quad D(J) \subseteq D(A)$$
$$\text{(ii)} \quad AJ \subseteq JA$$

where J is the involution associated with p.

It follows from condition (ii) in Proposition 1 that $H^{\pm}$ are closed. Consequently a reducible A has the representation

$$A = \begin{pmatrix} A^{+} & 0 \\ 0 & A^{-} \end{pmatrix} \qquad A^{\pm} \in L(u^{\pm})$$

Note that A can be unbounded although A^{+} and A^{-} are bounded. A non-vanishing reducible selfadjoint operator A can not be nilpotent.

Theorem 5 If A and $A^{\#}$ are reduced by a quasi-decomposition norm p, then A is p-continuous.

Suppose A is selfadjoint and reducible. The bounded operator A^{+} on H^{+} is $\langle\cdot|\cdot\rangle$-selfadjoint and according to

Theorem 5 also p-bounded. There is a positive operator $a^+ \in L(H^+)$ such that $<x|y> = (a^+x|y)$ $\forall x,y \in H^+$. The mapping

$$H^+ \ni x \to (a^+)^{\frac{1}{2}} x$$

extends to a unitary U^+ mapping the p-completion $(\overline{H^+}, p)$ of H^+ onto $(H^+, (\cdot|\cdot))$. We set

$$\hat{A}^+ = U^+\overline{A^+}(U^+)^*$$

where $\overline{A^+}$ denotes the extension of A^+ to $(\overline{H^+}, p)$ and observe that $\hat{A}^+$ is a bounded selfadjoint operator on the Hilbert space $(H^+, (\cdot|\cdot))$. Consequently $\hat{A}^+$ allows a spectral decomposition

$$\hat{A}^+ = \int \lambda d\hat{E}^+(\lambda)$$

If a spectral decomposition of A^+ in terms of $<\cdot|\cdot>$-selfadjoint projections should exist, the spectral measure $E^+(\lambda)$ must satisfy

$$\hat{E}^+(\lambda) = U^+\overline{E^+(\lambda)}(U^+)^*$$

However, this equation has in general no operator solutions. Thus a spectral theory in terms of operator distributions must be adopted or conditions on A ensuring operator solutions to the above equation must be imposed.

<u>Theorem 6</u>

If $A = \begin{pmatrix} A^+ & 0 \\ 0 & 0 \end{pmatrix}$ is a reducible generator for a semigroup of bounded transformations on H, then A is bounded.

Reference

[1] F. Hansen: Selfpolar Norms on an Indefinite Inner Product Space. RIMS-262 September 1978 (preprint).