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TWO PROBLEMS ON ORDERABLE SEMIGROUPS

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I. A semigroup S is said to be an orderable semigroup or an o-semigroup if S admits a simple order to make it a simply ordered semigroup.

PROBLEM 1. Characterise right cancellative, right simple o-semigroups without idempotents. (This problem was proposed in our lecture note [6].)

In connection with the above problem we have the following two results.

RESULT 1. Let p, q be two infinite cardinals such that $q \leqq p$ and let $S(p,q)$ be a Baer-Levi semigroup of type (p,q). Then $S(p,q)$ is a right cancellative, right simple semigroup without idempotents but is not an o-semigroup.

The first assertion is given in [1] Theorem 8.2. Now by way of contradiction, we assume that $S(p,q)$ is an o-semigroup. Thus $S(p,q)$ can be considered as a simply ordered semigroup.

First suppose $p = q$. By definition, there exists a set A such that $|A| = p$ and $S(p.p)$ is the family of all injective mappings α of A into A with $|A \setminus \alpha A| = p$. Let B_1, B_2, B_3 be mutually disjoint subsets of A such that $|B_1| = |B_2| = |B_3| = p$ and $B_1 \cup B_2 \cup B_3 = A$. Then for $i = 1, 2, 3$, there exists an injective mapping α_i of A onto B_i. Without loss of generality, we assume $\alpha_1 \leqq \alpha_2 \leqq \alpha_3$ in the simply ordered semigroup $S(p,p)$.

Since $S(p,p)$ is simply ordered without idempotents, we have either $\alpha_2 < \alpha_2^2$ or $\alpha_2^2 < \alpha_2$. Suppose $\alpha_2 < \alpha_2^2$. Then, since $S(p,p)$ has no idempotents, it follows from [5] Lemma 2 that we have $\alpha_3 < \alpha_3^2$. We have $A\alpha_2^2 = B_2\alpha_2 \subseteq A\alpha_2 = B_2$ and $A\alpha_3^2 = B_3\alpha_3 \subseteq A\alpha_3 = B_3$. Moreover

$$p = |B_1| \leqq |B_1 \cup (B_2 \cup A\alpha_2^2)| \leqq |A| = p,$$
$$p = |B_1| \leqq |B_1 \cup (B_3 \cup A\alpha_3^2)| \leqq |A| = p,$$

and so $|B_1 \cup (B_2 \cup A\alpha_2^2)| = |B_1 \cup (B_3 \cup A\alpha_3^2)| = p$. Since p is an infinite cardinal, we can choose a mutually dosjoint sets C and D such that $C \cup D = B_1 \cup (B_2 \cup A\alpha_2^2)$ and $|C| = |D| = p$. Since $|B_1 \cup (B_3 \cup A\alpha_3^2)| = p = |C|$, there exists an injection γ of $B_1 \cup (B_3 \cup A\alpha_3^2)$ onto C. Now we define a mapping β by:

$$x\beta = \begin{cases} x\alpha_3^{-1} & \text{if } x \in A\alpha_3^2, \\ x\alpha_2 & \text{if } x \in B_2, \\ x\gamma & \text{if } x \in B_1 \cup (B_3 \cup A\alpha_3^2). \end{cases}$$

Then β is a injection of A into A and

$$|A \setminus A\beta| = |A \setminus (A\alpha_3 \cup B_2\alpha_2 \cup C)| = |D| = p$$

and so $\beta \in S(p,p)$. Moreover, for every $x \in A$, we have $x\alpha_2 \in A\alpha_2 = B_2$ and $x\alpha_3^2 \in A\alpha_3^2$ and so $x\alpha_2\beta = x\alpha_2^2$ and $x\alpha_3^2\beta = x\alpha_3^2\alpha_3^{-1} = x\alpha_3$. Hence $\alpha_2\beta = \alpha_2^2$ and $\alpha_3^2\beta = \alpha_3$. Since $\alpha_2 < \alpha_2^2$, we have $\alpha_2^2 \leqq \alpha_2^3$ and, since $S(p,p)$ has no idempotents, we have $\alpha_2^2 < \alpha_2^3$. Hence

$$\alpha_2^2 < \alpha_2^3 = \alpha_2\alpha_2^2 = \alpha_2(\alpha_2\beta) = \alpha_2^2\beta = (\alpha_2\beta)\beta = \alpha_2\beta^2$$

and so $\alpha_2 < \beta^2$. Hence by [5] Lemma 2, we have $\beta^2 < (\beta^2)^2 = \beta^4$. But $\beta^2 \leqq \beta$ would imply that $\beta^4 \leqq \beta^3 \leqq \beta^2$. Since $S(p,p)$ is simply ordered, we have $\beta < \beta^2$. Hence

$$\alpha_3\beta \leqq \alpha_3^2\beta \leqq \alpha_3^2\beta^2 = (\alpha_3^2\beta)\beta = \alpha_3\beta$$

and so $\alpha_3\beta = \alpha_3^2\beta$. Hence

$$\alpha_3 = \alpha_3^2\beta = \alpha_3(\alpha_3\beta) = \alpha_3(\alpha_3^2\beta) = \alpha_3^3\beta = \alpha_3(\alpha_3^2\beta) = \alpha_3^2,$$

which contradicts the assumption that $S(p,p)$ has no idempotents.

In the case where $\alpha_2^2 < \alpha_2$, we can deduce a contradiction in a similar way.

Next we consider a general $S(p,q)$. We take an arbitrary $\alpha \varepsilon S(p,q)$ and put $T = \{ \xi \varepsilon S(p,q);\ \alpha\xi = \alpha \}$. Since $S(p,q)$ is right simple, we have $\alpha S = S$ and so T is nonempty. If $\xi, \eta \varepsilon T$, then $\alpha(\xi\eta) = (\alpha\xi)\eta = \alpha\eta = \alpha$, $\xi\eta \varepsilon T$ and so T is a subsemigroup of $S(p,q)$. Since $\alpha \varepsilon S(p,q)$, α is an injection of a set A into A such that $|A| = p$ and $|A \setminus A\alpha| = q$. Also for $\xi \varepsilon S(p,q)$, $\xi \varepsilon T$ if and only if ξ induces the identity mapping on $A\alpha$. For each $\xi \varepsilon T$, we denote by $\bar{\xi}$ the restriction of ξ to $A \setminus A\alpha$. Since ξ is an injection of A into A which induces the identity mapping on $A\alpha$, $\bar{\xi}$ is an injection of $A \setminus A\alpha$ into $A \setminus A\alpha$. Moreover, since $|A \setminus A\alpha| = q$ and

$$|(A \setminus A\alpha) \setminus (A \setminus A\alpha)\xi| = |A \setminus A\xi| = q,$$

$\bar{T} = \{ \bar{\xi};\ \xi \varepsilon T \}$ is a Baer-Levi semigroup $S(q,q)$. Further the mapping of T onto $\bar{T}$ which maps ξ into $\bar{\xi}$ is an isomorphism of T onto $\bar{T}$. Now since $S(p,q)$ is an o-semigroup, the subsemigroup T of $S(p,q)$ is also an o-semigroup. Hence $\bar{T} = S(q,q)$ is an o-semigroup, which contradicts the fact proved above.

<u>RESULT 2</u>. <u>There really exists a right cancellative, right simple o-semigroup without idempotents</u>.

In fact, let S be the set of all realvalued continuous functions α defined on the closed interval $[0,1]$, satisfying the conditions that $0 < 0\alpha$, $1\alpha < 1$ and the graph of α can be represented by a finite number of strictly increasing segments. It can be proved that S is a semigroup under the operation of composite of mappings and the semigroup S is right cancellative, right simple and has no idempotents (cf. [3]). Also it can be shown that S is a simply ordered semigroup under the order defined by:

for $\alpha, \beta \in S$, $\alpha < \beta$ if and only if there exist real numbers c and δ such that $0 \leqq c < 1$, $\delta > 0$, $x\alpha = x\beta$ for every $0 \leqq x < c$ but $x\alpha < x\beta$ for every $c < x < c + \delta$.

II. RESULT 3. *The collection of all idempotent o-semigroups does not form a variety.*

In fact, let L be a left zero semigroup and let R be a right zero semigroup. Then it can be checked that, with respect to an arbitrary simple order on L, L is a simply ordered semigroup and, with respect to an arbitrary simple order on R, R is a simply ordered semigroup. Hence L and R are o-semigroups. In particular, if $|L| \geqq 2$ and $|R| \geqq 2$ and if S is the direct product semigroup of L and R, then S is a rectangular band which is neither a left zero semigroup nor a right zero semigroup. Hence by [4] Theorem 1, S is not an o-semigroup. Hence the collection of all idempotent o-semigroups is not closed with respect to the formation of direct products and so is not a variety.

Since the intersection of a family of varieties of semigroups is a variety of semigroups, we can consider a variety of semigroups which is generated by idempotent o-semigroups.

In connection with this, we give the following problem.

PROBLEM 2. *Give the concrete description of the variety of semigroups generated by idempotent o-semigroups.*

REFERENCES

1. A. H. Clifford and G. B. Preston, The algebraic theory of semigroups vol.II, Amer. Math. Soc., 1967.

2. U. K. Kuspanov, On the classes of orderable semigroups in varieties, Semigroup Forum 16 (1978), 117-131.

3. T. Saitô, An example of left simple semigroup, Bull. Tokyo Gakugei Univ., 10 (1959), 139-142.

4. T. Saitô, Ordered idempotent semigroups, J. Math. Soc. Japan, 14 (1962), 150-169.

5. T. Saitô, Regular elements in an ordered semigroup, Pacific J. Math. 13 (1963), 263-295.

6. T. Saitô, Cours sur les demi-groupes totalement ordonnes, Université de Paris VI, 1972.

7. T. Saitô, The orderability of idempotent semigroups, Semigroup Forum 7 (1974), 264-285.

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