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# A Move Problem on Weighted Digraphs

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## 1. Definitions

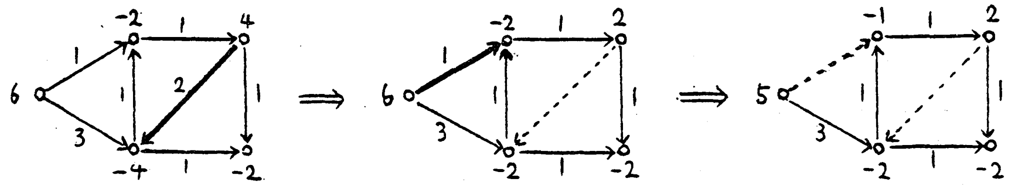
Let's consider a weighted digraph $\mathscr{D} = (D, \alpha, \beta)$ where D is a simple digraph, $\alpha$ is a function from the vertex-set V(D) of D to the set Z of integers and $\beta$ is a function from the arc-set A(D) of D to the set I of non-negative integers. The weighted digraph $\mathscr{D}$ is called a <u>w-digraph</u>. A <u>move</u> t in a w-digraph $\mathscr{D} = (D, \alpha, \beta)$ is defined by a reassignment to an arc (x,y) with the end vertices x and y in $\mathscr{D}$ such that

$$\begin{array}{ll} x & \alpha(x) \\ \downarrow & \beta(x,y) \\ y & \alpha(y) \end{array} \quad \xrightarrow{t} \quad \begin{array}{ll} x & \alpha(x) - k \\ \downarrow & \beta(x,y) - k \\ y & \alpha(y) + k \end{array}$$

where $0 \leq k \leq \beta(x,y)$. If k = 0 then the move t is called an <u>empty move</u>. If k = 1 then the move t is called a <u>unit move</u>. A w-digraph resulted from a move t in a w-digraph $\mathscr{D}$ is called a <u>resultant of $\mathscr{D}$ by</u> t, written $t(\mathscr{D})$. A sequence of moves starting from a move in a w-digraph $\mathscr{D}$ is called a <u>move process from $\mathscr{D}$</u>. A w-digraph resulted from a move process $\mathbf{t}$ from $\mathscr{D}$ is called a <u>resultant of $\mathscr{D}$ by</u> $\mathbf{t}$, written $\mathbf{t}(\mathscr{D}) = (D, \mathbf{t}(\alpha), \mathbf{t}(\beta))$.

For any w-digraphs $\mathcal{D}_1$ and $\mathcal{D}_2$, a move process $\mathbf{t}$ such that $\mathbf{t}(\mathcal{D}_1) = \mathcal{D}_2$ is called a <u>move process from $\mathcal{D}_1$ to $\mathcal{D}_2$</u>. Note that $\mathbf{t}(\mathcal{D}) = \mathbf{t}'(t(\mathcal{D}))$ for a move process $\mathbf{t} = \mathbf{t}'t$ from $\mathcal{D}$, where t is a move from $\mathcal{D}$.

<u>Example</u> 1. A move process is illustrated as follows.




2. Problems

The following problems were proposed by the author in 1975. Let a digraph D, three functions $\alpha_0 : V(D) \longrightarrow Z$, $\alpha_1 : V(D) \longrightarrow Z$ and $\beta_0 : A(D) \longrightarrow I$ be given. Then, determine (by a nontrivial algorithm) whether or not there is a move process from $\mathcal{D}_0 = (D, \alpha_0, \beta_0)$ to $\mathcal{D}_1 = (D, \alpha_1, \mathbf{t}(\beta_0))$. The problem is called a <u>move problem</u>. For a w-digraph $\mathcal{D} = (D, \alpha, \beta)$, we denote $\sum_{x \in V(D)} |\alpha(x)|$ by $\tilde{\alpha}_{\mathcal{D}}$. For a w-digraph $\mathcal{D}$ and a move process $\mathbf{t}$ from $\mathcal{D}$, $\tilde{\alpha}_{\mathcal{D}} - \tilde{\alpha}_{\mathbf{t}(\mathcal{D})}$ is called a <u>matching quantity of $\mathcal{D}$ by $\mathbf{t}$</u>, written $m(\mathcal{D}, \mathbf{t})$. A move process $\mathbf{t}$ from $\mathcal{D}$ such that $m(\mathcal{D}, \mathbf{t})$ takes a maximum is called a <u>maximum matching</u> in $\mathcal{D}$ and a move process $\mathbf{t}$ from $\mathcal{D}$ such that $\tilde{\alpha}_{\mathbf{t}(\mathcal{D})} = 0$ is called a <u>complete matching</u> in $\mathcal{D}$. Note that a maximum matching $\mathbf{t}$ in $\mathcal{D}$ is a complete matching in $\mathcal{D}$ if $m(\mathcal{D}, \mathbf{t}) = \tilde{\alpha}_{\mathcal{D}}$. Now, we call " to give a nontrivial algorithm for obtaining a maximum matching in a given w-digraph" a <u>maximum matching problem</u> and "to determine (by a nontrivial algorithm) whether or not there is a complete matching in a given w-digraph" a <u>complete matching problem</u>. We also call the well-known problem to construct a maximum flow of a given network " the maximum flow problem".

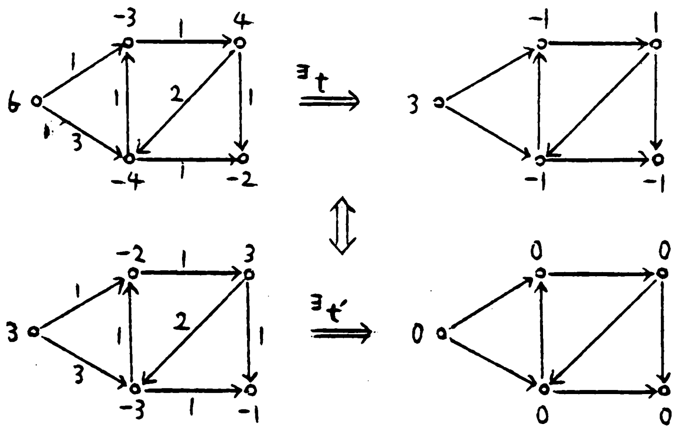
3. Propositions

The following results have been obtained by the author, Yukio Kusaka and Hajime Sato.

Proposition 1 (H. Narushima). The move problem is equivalent to the complete matching problem.

Proof. It is easily verified that there is a move process **t** such that $\mathbf{t}((D, \alpha_0, \beta_0)) = (D, \alpha_1, \mathbf{t}(\beta_0))$ if and only if there is a move process **t'** such that $\mathbf{t}'((D, \alpha_0 - \alpha_1, \beta_0) = (D, \alpha = 0, \mathbf{t}'(\beta_0))$.

Example 2. The proof of Prop. 1 is illustrated as follows.



Proposition 2 (Y. Kusaka, H. Sato and H. Narushima). The maximum matching problem on w-digraphs is equivalent to the maximum flow problem on networks.

Proof. We first construct a network $N = (D_N, \beta_N)$ from a given w-digraph $\mathcal{D} = (D, \alpha, \beta)$ as follows. Let

$$V^+ = \{v \in V(D) \mid \alpha(v) > 0\}, \quad V^- = \{v \in V(D) \mid \alpha(v) < 0\}.$$

Then $V(D_N)$, $A(D_N)$ and $\beta_N$ are defined :

$$V(D_N) = V(D) \cup \{\oplus, \ominus\}, \text{ where } \oplus, \ominus \notin V(D),$$

$$A(D_N) = A(D) \cup \{(\oplus, v) \mid v \in V^+\} \cup \{(v, \ominus) \mid v \in V^-\},$$

$$\beta_N(a) = \begin{cases} \beta(a) & \text{for } a \in A(D) \\ |\alpha(v)| & \text{for } a = (\oplus, v) \text{ or } (\ominus, v). \end{cases}$$

Conversely, we construct a w-digraph $\mathcal{D} = (D_w, \alpha_w, \beta_w)$ from a given network $N = (D, \beta)$ as follows. Let the source in N and the sink be denoted by $\oplus$ and $\ominus$, respectively. Let

$$V^+ = \{v \in V(D) \mid (\oplus, v) \in A(D)\}, \; V^- = \{v \in V(D) \mid (v, \ominus) \in A(D)\}.$$

Then $D_w$, $\alpha_w$ and $\beta_w$ are defined:

$D_w = D - \{\oplus, \ominus\}$, where the arcs incident to $\oplus$ or $\ominus$ are removed,

$$\alpha_w(v) = \begin{cases} \beta(\oplus, v) & \text{for } v \in V^+ \\ -\beta(v, \ominus) & \text{for } v \in V^- \\ 0 & \text{otherwise,} \end{cases}$$

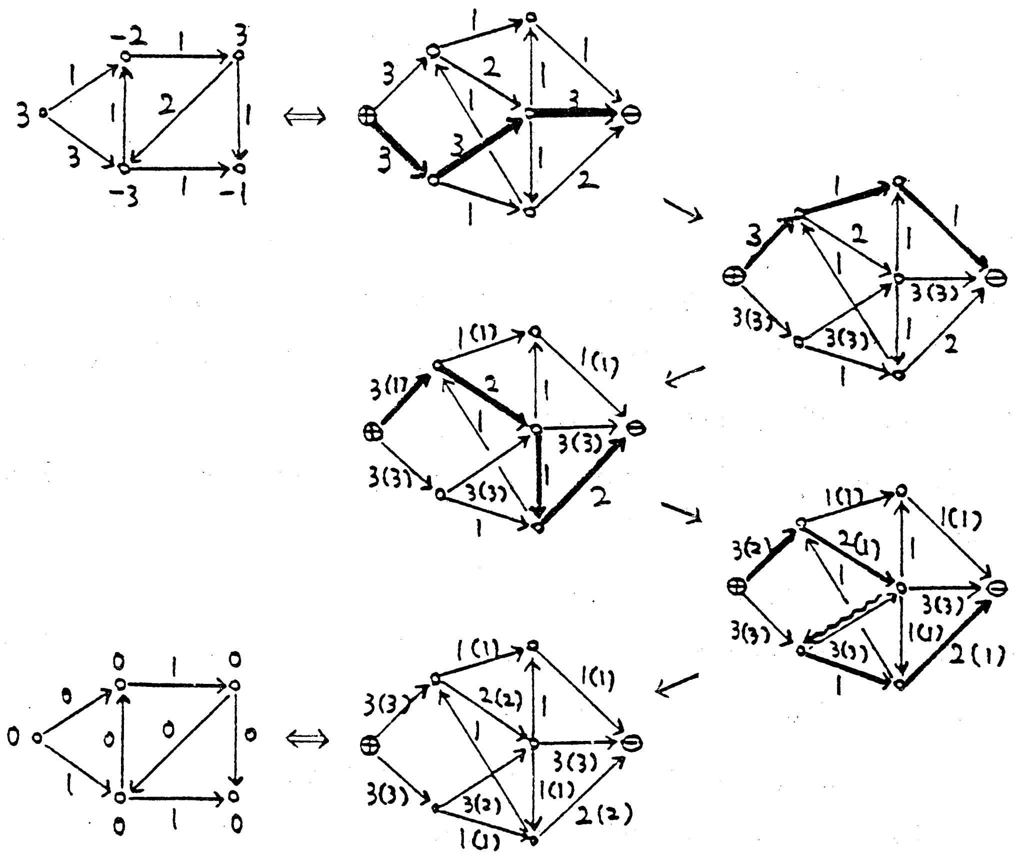
$\beta_w = \beta|_{A(D_w)}$ ( the restriction of $\beta$ to $A(D_w)$ ).

The remainder of the proof are easy.

<u>Proposition</u> 3 (Y. Kusaka, H. Sato and H. Narushima). The move problem, the complete matching problem and the maximum matching problem are all solvable by a nontrivial algorithm.

<u>Proof</u>. The maximum flow probelm on networks are solvable by the well-known labelling method and therefore, by Prop. 2 the maximum matching problem on w-digraphs are solvable. As it is easily judged whether or not a maximum matching $\mathfrak{t}$ in $\mathcal{D}$ is a complete matching, i.e., the criterion is $m(\mathcal{D}, \mathfrak{t}) = \tilde{\alpha}_{\mathcal{D}}$, the complete matching problem on w-digraphs are solvable. Therefore, by Prop. 1 the move problem are solvable.

Example 3. Prop. 3 is demonstrated as follows.



4. Applications

An application of our formulation and results for recitation schedulings is given in [1, 2]. The program in [1] is a FORTRAN program (made by H. Mori) of which size is composed of about 400 steps. It is the main idea of the algorithm to decrease as much as possible the difference between a maximal value of a vertex and a minimal value of a vertex, by a process of unit moves, with a consideration of the connectivity of the w-digraphs. The connectivity consideration means "not to increase as much as possible the number of connected components of a given w-digraph". Note that there is a move process **t** such that $\mathbf{t}(\mathcal{D}_0) = \mathcal{D}_1$ but neverthless it happens that the algorithm fails in reaching $\mathcal{D}_1$ from $\mathcal{D}_0$. The program in [2] is a PL/I program (made by Y. Kusaka) of which size is composed of about 80 steps. The algorithm is based on Prop. 2. Note that if we use the algorithm, it happens that the remainder of weights concentrate into some vertices when a maximum matching is not a complete matching. The author thinks that his formulation of w-digraphs is very useful for many kinds of OR problems in System Dynamics.

References

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