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Semigroup theory for functional differential equations with infinite delay; a representation of infinitesimal generators

Toshiki Naito
The University of Electro-Communications

1. Semigroups and infinitesimal generators. Suppose $\mathcal{B} = \mathcal{B}(I, C^n)$ is a linear space of some functions ϕ mapping an intermal I into n-dimensional comlex linear space C^n, where $I = [-r, 0]$, $0 \leqq r < \infty$, or $I = (-\infty, 0]$. For a C^n-valued given function x and a parameter t in R, the function $x_t : I \to C^n$ is defined by $x_t(\theta) = x(t+\theta)$ for θ in I whenever $x(t+\theta)$ is well defined. If $L : \mathcal{B} \to C^n$ is a given linear operator, we say that the relation

$$(1) \qquad x'(t) = L(x_t)$$

is a linear functional differential equation--with finite delay when $I = [-r, 0]$, or with infinite delay when $I = (-\infty, 0]$. In this lecture, L is always assumed to be continuous. Suppose for every ϕ in $\mathcal{B}$ Equation (1) has a unique solution $x(t;\phi)$ for t in $[0, \infty)$ with the initial condition $x_0 = \phi$. Then the solution operator $T(t) : \mathcal{B} \to \mathcal{B}$ is defined by the relation

$$T(t)\phi = x_t(\phi), \quad \phi \text{ in } \mathcal{B}, \quad t \geqq 0.$$

In case $\mathcal{B}$ is the family of continuous functions on $[-r, 0]$ into C^n, the family $\{T(t): t \geqq 0\}$ is a semigroup of class (C_0) of bounded linear operators on $\mathcal{B}$. Let A be the infinitesimal generator of $T(t)$; that is, $A\phi = \lim_{t\to 0+} t^{-1}[T(t)\phi - \phi]$ whenever this limit exists. It is well known [2] that A is given by

(2) $$A\phi(\theta) = \begin{cases} L(\phi) & \text{for } \theta = 0 \\ \phi'(\theta) & \text{for } -r \leqq \theta < 0 \end{cases}$$

if and only if the function defined by the relation in the right hand side belongs to $\mathcal{B}$.

In the case when $\mathcal{B} = \mathcal{B}((-\infty, 0], C^n)$, several models for $\mathcal{B}$ are proposed: for some special measure μ, $\mathcal{B} = L^p(\mu) \times C^n$ in which a norm is defined by $|\phi| = [|\phi(0)|^p + \int_{-\infty}^0 |\phi(\theta)|^p \, d\mu(\theta)]^{1/p}$, $1 \leqq p < \infty$; $\mathcal{B} = C_\gamma$, the family of continuous functions ϕ such that $\phi(\theta)\, e^{\gamma\theta} \to$ a limit as $\theta \to -\infty$, in which a norm is defined by $|\phi| = \sup |\phi(\theta)e^{\gamma\theta}|$. In these cases the family $T(t)$ is again a semigroup of class (C_0) of bounded linear operators on $\mathcal{B}$. Furthermore, the representation of A similar to Formula (2) is valid; the definition of ϕ' is slightly changed according to the choice of $\mathcal{B}$ (cf [3, 4]).

In [2, 5, 6], etc, Equation (1) is considered on some abstract phase space $\mathcal{B}$ which is defined to be a space satisfying some hypotheses. The systems of hypotheses are somewhat different from each other according to the problem under discussion. In all cases, however, $T(t)$ becomes a semigroup of bounded linear operators on $\mathcal{B}$. Further results are known [5, 6]: an asymptotic estimate of the order of $|T(t)|$ as $t \to \infty$; informations about the spectrum of A; and a construction of the fundamental matrix of Equation (1) with the variation-of-constants formula for the forced system of Equation (1), etc. However, it has been left unsolved to represent A in the manner analogous to Formula (2).

2. <u>Formal approaches to the problem.</u> To explain the reason why the representation of A is difficult to obtain, we go into the details of the hypotheses on $\mathcal{B}$ employed in [5, 6].

(H-0). A seminorm $|\cdot|$ is defined on $\mathcal{B}$: the quotient space $\hat{\mathcal{B}} = \mathcal{B}/|\cdot|$ is a Banach space.

(H-1). If a function $x:(-\infty, \sigma+\alpha) \to C^n$, $\alpha > 0$, is continuous on $[\sigma, \sigma+\alpha)$ and x_σ is in $\mathcal{B}$, then x_t is in $\mathcal{B}$ for every t in $[\sigma, \sigma+\alpha)$ and the map $t \to x_t$ is continuous.

(H-2). There exist positive continuous functions $K(t)$ and $M(t)$, where $M(t)$ is submultiplicative, such that, for the function x arising in (H-2), $|x_t| \leqq K(t-\sigma) \times \sup\{|x(s)|:\sigma \leqq s \leqq t\} + M(t-\sigma)\ |x_\sigma|$ for $\sigma \leqq t < \sigma+\alpha$.

(H-3). $|\phi(0)| \leqq K|\phi|$, ϕ in $\mathcal{B}$, for some constant K.

From these hypotheses, it follows that the solution $x(t;\phi)$ exists on $[0, \infty)$ uniquely: the solution operator $T(t)$ is linear and continuous on $\mathcal{B}$. Hypothesis (H-1) implies that the semigroup $T(t)$ is of class (C_0). However, notice that no measurability condition is assumed on ϕ in $\mathcal{B}$. We cannot, for example, discuss whether ϕ in $\mathcal{B}$ is absolutely continuous or not: the derivative ϕ' has no meaning. To overcome this difficulty, we add more hypotheses on $\mathcal{B}$, or else we interpret Formula (2) in a different manner than ever before. In this lecture, we proceed along the latter line.

To do this, let us introduce operators B and C_L defined formally by the relations

$$B\phi(\theta) = \begin{cases} 0 & \theta = 0 \\ \phi'(\theta) & \theta < 0 \end{cases} \qquad C_L\phi(\theta) = \begin{cases} L(\phi) & \theta = 0 \\ 0 & \theta < 0. \end{cases}$$

To emphasize the operator $L:\mathcal{B} \to C^n$, $T_L(t)$ denotes the solution semigroup of Equation (1) and A_L its infinitesimal generator. Then we can rewrite Formula (2) as

$$A_L\phi = B\phi + C_L\phi. \tag{3}$$

Observe that $B = A_0$, the infinitesimal generator of the solution semigroup $T_0(t)$ of the trivial equation $x'(t) = 0$. Usually, we use a special symbol $S(t)$ for $T_0(t)$. The operator $U_L(t)$ defined by $U_L(t) = T_L(t) - S(t)$ is completely continuous; the decomposition $T_L(t) = S(t) + U_L(t)$ was extremely useful to investigate the property of $T_L(t)$, [3,5]. Hence Relation (3) is also expected to have some meaning. However, we soon notice that this formula contains a trivial contradiction; that is, the domains of A_L and B do not coincide with each other. Furthermore, we do not know whether C_L is well defined on the space $\mathcal{B}$ or not. Formula (3) has an ambiguity concerning the domain where it holds.

3. Representation of A in the dual space. Fortunately, the adjoint operators of A_L and B have the same domain. Stech [7] first discovered this interesting fact in the case where $\mathcal{B}$ is of the type $L^p(\mu) \times C^n$; the author [5] proved the same result in the case where $\mathcal{B}$ is an abstract space satisfying a system of hypotheses similar to (H-0, ... 4). Therefore, we hope that Formula (3) can be interpreted if the relation is transferred to the dual space. Let us introduce notations: X^* is a dual space of a Banach space X, and T^* the dual operator of a linear operator T on X if it exists.

Before the demonstration of the final result, we again refer to the hypotheses on $\mathcal{B}$ which is sufficient to obtain the desired result. We leave Hypotheses (H-0) and (H-1) as they are. The latter hypothesis implies that $\mathcal{B}$ contains the space $\mathcal{C}$, the family of continuous functions on $(-\infty, 0]$ with compact supports. Hypothesis (H-2) is replaced by

(H-2)'. There exists a continuous function $K(t)$ such that, if ϕ in $\mathcal{C}$ has its support in $[-t, 0]$, then

$$|\phi| \leqq K(t) \sup\{|\phi(\theta)|: -t \leqq \theta \leqq 0\}.$$

Suppose $\alpha:\mathcal{B} \to C$ is linear and continuous; that is, α is a member of $\mathcal{B}^*$. Then (H-2)' implies

$$|\langle\alpha, \phi\rangle| \leqq |\alpha| K(t) \sup\{|\phi(\theta)|: -t \leqq \theta \leqq 0\}$$

for ϕ arising in (H-2)'. This means that the restriction of α on $\mathcal{C}$ is a Radon measure on $(-\infty, 0]$. It is well known that, for such a measure α, there exists a function $\eta(\alpha;\theta) = (\eta_1(\alpha;\theta), \ldots, \eta_n(\alpha;\theta))$ locally of bounded variation for θ on $(-\infty, 0]$ such that

$$\langle\alpha, \phi\rangle = \int_{-\infty}^{0} d_\theta\eta(\alpha;\theta)\, \phi(\theta) = \int_{-\infty}^{0} \sum_{i=1}^{n} d_\theta\eta_i(\alpha;\theta)\, \phi^i(\theta)$$

for every ϕ in $\mathcal{C}$. We can assume that η is normalized in the sense that $\eta(\alpha;0) = 0$ and $\eta(\alpha;\theta)$ is continuous to the left for $\theta < 0$. Then $\eta(\alpha;\theta)$ is determined uniquely by α in $\mathcal{B}^*$. It is clear that the map $\alpha \to \eta(\alpha;0-)$, the left-hand limit of η at $\theta = 0$, is a linear operator on

$\mathcal{B}^*$ into C^n. We assume the following hypothesis.

(H-4). This operator $\alpha \to \eta(\alpha;0-)$ is continuous.
Note that this condition holds [6] if we adds one more hypothesis to the system (H-0, 1, 2, 3).

Hypothesis (H-3) is also assumed and rewritten in the form

(H-3). The operator $D:\mathcal{B} \to C^n$ defined by $D(\phi) = \phi(0)$, $\phi \in \mathcal{B}$, is continuous.

Finally, we need the following.

(H-5). For every $t \geqq 0$, $T_L(t)$ is well defined to be a continuous linear operator on $\mathcal{B}$.
Hypotheses (H-1, 5) imply that $T_L(t)$ is a semigroup of class (C_0) of bounded linear operators on $\mathcal{B}$. It is known [2, 5] that (H-5) is derived from (H-1,2,3). In this lecture, we are not interested in this fact, but we dovote ourself to the study of the representation of A_L. From this standpoint, we assume the above statement, while (H-2) is replaced by a weak Hypothesis (H-2)'.

Now observe that $(T_L(t) - S(t))\phi$ is a menber of C for every ϕ in $\mathcal{B}$. From Hypothesis (H-2)', for every α in $\mathcal{B}^*$ we can represent $\langle\alpha, (T_L(t) - S(t))\phi\rangle$ in terms of Stirtjes integral. This implies the following Proposition.

<u>Proposition 1.</u> For every α in $\mathcal{B}^*$ and every ϕ in $\mathcal{B}$,

$$\lim_{t\to 0+} \langle t^{-1}[T_L^*(t) - S^*(t)]\alpha, \phi\rangle = \sum_{i=1}^{n} -\eta_i(\alpha;0-)\, L^i(\phi).$$

Let $\delta^i(\phi)$ and $L^i(\phi)$ denote the i-th component of $D(\phi)$ and $L(\phi)$, respectively. Hypothesis (H-3) implies that δ^i is an element of $\mathcal{B}^*$; clearly, L^i is also in $\mathcal{B}^*$. From the definition of infinitesimal generators and dual operators, we have the following proposition.

Proposition 2. Every δ^i belongs to the domain of A_L^*; and $A_L^*\delta^i = L^i$ for $i = 1, \ldots, n$.

Define an operator $P:\mathcal{B}^* \to \mathcal{B}^*$ by the relation

$$P\alpha = -\eta(\alpha;0-)\cdot D = \sum_{i=1}^{n} -\eta_i(\alpha;0-)\delta^i \quad \text{for } \alpha \text{ in } \mathcal{B}^*.$$

It is easy to see that $PP = P$; while Hypothesis (H-4) implies that P is continuous. Therefore P is a continuous projection on $\mathcal{B}^*$. Now we state the main theorem.

Theorem 3. Suppose Hypotheses (H-0,1,3,4,5) and (H-2)' hold and let A_L be the infinitesimal generator of the solution semigroup $T_L(t)$ of Equation (1). Then the domain of A_L^* is independent of the choice of the continuous linear operator $L:\mathcal{B} \to C^n$. If the above projection P is restricted on this common domain $\mathcal{D}^*$, then the restriction, denoted by P again, is also a projection on $\mathcal{D}^*$. The all operators A_L^*, B^* and P are transformations of $\mathcal{D}^*$ into $\mathcal{B}^*$ and they are related with each other in the manner

$$A_L^* = B^* + A_L^*P.$$

It is not difficult to prove this theorem if we use Proposition 1, 2 and the following result (cf. [1, p. 49]): the dual operator of the infinitesimal generator of a semigroup of class (C_0) is equal to the weak* infinitesimal generator of the dual semigroup. Notice thet, along this line, we can again prove the existence of the common domain $\mathcal{D}^*$.

Corollary 4. The domain $\mathcal{D}^*$ is decomposed into a direct sum $\mathcal{D}^* = \mathcal{N}_L^* \oplus \mathcal{M}_L^*$ as follows:
(i) $A_L^*\alpha = B^*\alpha$ if and only if α is in $\mathcal{N}_L^*$.
(ii) The restriction of A_L^* on $\mathcal{M}_L^*$ is an isomorphism of $\mathcal{M}_L^*$ onto the linear manifold generated by $\{L^1, \ldots, L^n\}$.
(iii) $\mathcal{M}_L^*$ is contained in $P\mathcal{D}^*$ and $(I - P)\mathcal{D}^*$ in $\mathcal{N}_L^*$.
On the other hand, the following conditions are equivalent:
(a) $\mathcal{M}_L^* = P\mathcal{D}^*$, (b) $\mathcal{N}_L^* = (I - P)\mathcal{D}^*$, (c) the family $\{L^1, \ldots, L^n\}$ are linearly independent.

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