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Topics on Kac-Moody Lie algebras

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In this note we introduce Kac-Moody Lie algebras, their representation theory, and relation with theory of differential equations.

I. Kac-Moody Lie algebras ([6],[I4],[I5])

Let $C=(c_{ij})$ be a n×n-matix satisfying following conditions;

$c_{ij}\in\mathbb{Z}$, $c_{ii}=2$, $c_{ij}\leq 0$ $(i\neq j)$, $c_{ij}=0 \Leftrightarrow c_{ji}=0$,

and define Lie algebra $\mathcal{L}(C)$ over $\mathbb{C}$ with generators $\{h_i, e_i, f_i \ i=I,\cdots,n\}$ and relations;

$$\begin{cases} [h_i,h_j]=0 \ , \ [h_i,e_j]=c_{ij}e_j \ , \ [h_i,f_j]=-c_{ij}f_j \ , \ [e_i,f_j]=\delta_{ij}h_i \ , \\ (\mathrm{ad}\ e_i)^{-c_{ij}+I}e_j=0 \ , \ (\mathrm{ad}\ f_i)^{-c_{ij}+I}f_j=0 \ . \end{cases}$$

$\mathcal{L}=\mathcal{L}(C)$ is called Kac-Moody Lie algebra.

$\mathcal{L}$ has $\Gamma=\mathbb{Z}^n$ grading with deg $h_i=(0,\cdots,0)$, deg $e_i=(0,\cdots,\overset{i}{I},\cdots,0)=a_i$,and deg $f_i=-a_i$.

Put $\mathcal{L}_a=\{x\in\mathcal{L} \mid \deg x=a, (a\in\Gamma)\}$, $m_a=\dim\mathcal{L}_a$,$\Delta=\{a\in\Gamma\setminus\{0\} \mid m_a\neq 0\}$.

From defining relations $\mathcal{L}$ has vector space decomposition

$$\mathcal{L}=\mathcal{H}\oplus\mathcal{L}_+\oplus\mathcal{L}_- \text{ where } \mathcal{H}=\sum_i\mathbb{C}h_i \ , \mathcal{L}_+=\langle e_I,\cdots,e_n\rangle \ , \ \mathcal{L}_-=\langle f_I,\cdots,f_n\rangle \ .$$

So Δ is disjoint union of $\Delta_+=\Gamma_+\cap\Delta$ and $\Delta_-=(-\Gamma_+)\cap\Delta$

$(\Gamma_+=\mathbb{Z}_+a_I+\cdots+\mathbb{Z}_+a_n)$

For each i ,define s_i GL(Γ) by

$$s_i(a_j)=a_j-c_{ij}a_i$$

and $W=\langle s_i \mid i=I,\cdots,n\rangle$.

If C is decomposable, i.e. has a permutation matrix P such that

$$PCP^{-1}=\left(\begin{array}{c|c} C_1 & 0 \\ \hline 0 & C_2 \end{array}\right), \text{then } \mathcal{L}(C)\cong\mathcal{L}(C_1)+\mathcal{L}(C_2)\ ,\ W(C)\cong W(C_1)\times W(C_2).$$

So we assume C is indecomposable.

A finite-dimensional complex simple Lie algebra is isomorphic to a $\mathcal{L}(C)$ whose W(C) is finite group. Equivalent condition on C to finiteness of W(C) is that there exist positive numbers $d_1,\cdots,d_n$ such that $\begin{pmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{pmatrix}C$ is a positive definite symmetric matrix.

If we replace "positive definite" by "positive semi-definite" in this condition, we get a new class of infinite-dimensional Lie algebras. They are called Euclidean (or affine) Lie algebras. We deal with Euclidean case hereafter.

2. Character formula and denominator formula

Put $V=\mathbb{C}\otimes_{\mathbb{Z}}\Gamma$,and for $\lambda\in\mathfrak{h}^*$ let $c_\lambda: W\to V$ be the 1-cocycle, i.e. satisfies

$$c_\lambda(w_1w_2)= w_1c_\lambda(w_2) + c_\lambda(w_1)\ \text{,such that } c_\lambda(s_i)=\lambda(h_i)a_i \quad (i=1,\cdots,n).$$

If $\lambda(h_i)\in\mathbb{Z}$ for all i, λ is said integral, and moreover if $\lambda(h_i)\geq 0$, λ is said dominant integral. It is easy to see

$$\begin{cases} \lambda\text{ :integral} \Rightarrow c_\lambda(W)\subset\Gamma \\ \lambda\text{ :dominat integral} \Rightarrow c_\lambda(W)\subset\Gamma_+ \end{cases}$$

To each dominant integral, there exists $\mathcal{L}$ -module V^λ, unique up to isomorphism ,with the following property;

$$0\neq{}^{\exists}v\in V \text{ s.t. } \begin{cases} e_iv=0\ ,\ h_iv=\lambda(h_i)v\ ,\ f_i^{\lambda(h_i)}v=0 \quad (i=1,\cdots,n) \\ V=\sum\mathbb{C}f_{i_1}f_{i_2}\cdots f_{i_k}v \end{cases}$$

V^λ is called standard module with highest weight λ.

For $a\in\Gamma_+$,let $V_a^\lambda=\sum_{a_{i_1}+\cdots a_{i_k}=a}\mathbb{C}f_{i_1}\cdots f_{i_k}v$, and put

$$\operatorname{ch} V^\lambda=\sum_{a\in\Gamma_+}\dim V_a^\lambda e(-a) \quad (\operatorname{ch}V^\lambda \text{ is in } \mathbb{Z}[[e(-a_1),\cdots,e(-a_n)]].)$$

Let $\rho\in\mathfrak{h}^*$ be defined by $\rho(h_i)=I$ for all i.We have

character formula ([5],[8],[I0])

$$\mathrm{ch}\, V^{\lambda}=\frac{\sum_{w\in W}\det(w)e(-c_{\lambda+\rho}(w))}{\sum_{w\in W}\det(w)e(-c_{\rho}(w))}$$

and

denominator formula

$$\sum_{w\in W}\det(w)e(-c\ \ (w)) = \prod_{a\in\Delta_+}(I-e(-a))^{m_a}$$

If we make specialization $e(-a_i)\mapsto q^{m_i}$ $(m_I,\cdots,m_n\geq 0)$ to both sides of denominator (or character) formula,we have various combinatorial identities. ([7],[8],[I2],[I6])

3.K-dV equation and construction of representation

Let $u(x)$, $V_{n\ell}(x)$ $(n=I,2,3,\cdots,\ I\leq\ell\leq 2n-2)$ be functions in infinite many variables $x=(x_I,x_3,x_5,\cdots)$.From the compatibility conditions of linear partial differential equations

$$\left.\begin{array}{l}\left\{\left(\frac{\partial}{\partial x_1}\right)^2+u(x)\right\}\psi(x)=\lambda\psi(x)\\ \frac{\partial\psi}{\partial x_{2n-1}}=\left\{\left(\frac{\partial}{\partial x_1}\right)^{2n-1}+V_{nI}\left(\frac{\partial}{\partial x_1}\right)^{2n-3}+\cdots+V_{n\,2n-2}\right\}\psi\quad (n=I,2,\cdots)\end{array}\right\}$$

we obtain non-linear differential equations on u,and $V_{n\ell}$ are expressed by differential polynomials of u.These differential equations are called K-dV equations. A function $\tau(x)$ is called τ-function,if $u=\left(\frac{\partial}{\partial x_1}\right)^2\log\tau$ is a solution of K-dV equations.The space of infinitesimal trasformations of τ-functios forms a Lie algebra $\mathcal{L}$ of linear trans-formations on $C[x_I,x_3,\cdots]$. Date,Jimbo,Kashiwara and Miwa find ([I]) that $\mathcal{L}\simeq\mathcal{L}\begin{pmatrix}2&-2\\-2&2\end{pmatrix}$ and $\mathcal{L}$-module $C[x_I,x_3,\cdots]$ coincides with the standard module constructed by Lepowsky and Wilson.([II]). Kac,Kazhdan,Lepowsky and Wilson construct standard modules for other Euclidean Lie algebras in analogous way toL-W,and D-J-K-M show they correspond to some non-linear equations like K-dV.([2],[3] ,[9])

4. Remarks

(I) About classification and realizations of Euclidean Lie algebras,c.f. [6],[I5].

(2) Let $\mathfrak{R}\subset\mathcal{L}(C)$ be the maximam homogenious ideal with $\mathfrak{R}\cap\mathfrak{h}=0$ .Strictly speaking, $\mathfrak{g}(C)=\mathcal{L}(C)/\mathfrak{R}$ is Kac-Moody Lie algebra.But finite or Euclidean case $\mathfrak{R}$ =0,and we conjecture that $\mathfrak{R}$ =0 for any case.

(3) Character formula and denominator formula are analogy of finite case, and proved for $\mathfrak{g}$ (C) with symmetrizable C,i.e. there exist positive numbers $d_I, \cdots, d_n$ such that $\begin{pmatrix} d_I & & \\ & \ddots & \\ & & d_n \end{pmatrix}$C is symmetric matrix.

(4) Original K-dV equation is obtained by to put $x_5, x_7, \cdots$ constant.

(5) Frenkel and Kac ([4]) construct standard modules in different way from K-K-L-W.It will be interesting to describe isomorphism explicitly between K-K-L-W's module and F-K's.

References

[I] E.Date,M.Kashiwara and T.Miwa,Transformation groups for soliton equations II, RIMS preprint 357.

[2] E.Date,M.Jimbo,M.Kashiwara and T.Miwa, " IV,RIMS preprint 359 .

[3] " " " ", RIMS preprint 362.

[4] I.B.Frenkel and V.G.Kac,Basic representations of affine Lie algebras and dual resonance models,Inv.Math.,62(I980),23-66.

[5] H.Garland and J.Lepowsky,Lie algebra homology and the Macdonald-Kac formulas,Inv.Math.,34(I976),37-76.

[6] V.G.Kac,Simple irreducible graded Lie algebras of finite growth,Math.USSR Izvestija,2(I968),I27I-I3II.

[7] V.G.Kac,Infinite-dimensional Lie algebras and Dedekind's η -function,

Func.Analy.and its App.,8(1974),68-70.

[8] V.G.Kac,Infinite-dimensional algebras,Dedekind's η-function,classical Mobius function and the very strange formula,Adv.in Math.,30(1978),85-136.

[9] V.G.Kac,D.A.Kazhdan,J.Lepowsky and R.L.Wilson,Realization of the basic representations of the Euclidean Lie algebra,Adv.in Math.,to appear.

[10] J.Lepowsky,Lectures on Kac-Moody Lie algebras,Université Paris VI,1978.

[11] J.Lepowsky and R.L.Wilson,Construction of the affine Lie algebra $A_1^{(1)}$, Comm.Math.Physics,62(1978),43-53.

[12] J.Lepowsky and R.L.Wilson,A Lie theoretic interpretation and proof of Rogers-Ramanujan identities, preprint.

[13] I.G.Macdonald,Affine root systems and Dedekind's η-function,Inv.Math., 15(1972),91-143.

[14] R.V.Moody,A new class of Lie algebra,J.of Alg.,10(1968),211-230.

[15] ",Euclidean Lie algebra,Canad.J.Math.,21(1969),1432-1454

[16] " ,Macdonald identities and Euclidean Lie algebras,Proc.A.M.S. 48(1975),43-52.