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On separable extensions over a local ring

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1. Throughout this paper A is a ring with 1, and B is a subring of A such that $1 \in B$. A is a separable extension of B if and only if there exists $\Sigma x_i \otimes y_i$ in $A \otimes_B A$ such that $\Sigma x_i y_i = 1$ and $\Sigma x x_i \otimes y_i = \Sigma x_i \otimes y_i x$ for all x in A. For a ring automorphism σ of A we denote as usual by $A[X;\sigma]$ the ring of all polynomials with an indeterminant X whose multiplication is defined by $aX = X\sigma(a)$ for each a in A. Now consider the following condition

(T. P. R) $A = B[X;\sigma]/(X^2 - u)$ for some automorphism σ of B and a unit u of B such that $\sigma(u) = u$ and $au = u\sigma^2(a)$ for each a in B.

In the case where A and B satisfy the above condition, we will say that A and B satisfy condition (T. P. R) with σ. The aim of this paper is to show the following

Theorem. Let A be a separable extension of B, and suppose that $A = B \oplus M$ with a B-B-submodule M such that $M^2 \subset B$. Then A and B satisfy condition (T. P. R) with some automorphism σ of B, in the case where one of the following conditions is satisfied;

(1) B is a local ring, and M is finitely generated as left (or right) B-module

(2) B is a left (or right) Noetherian local ring

(3) B is a commutative Noetherian semi-local ring, and M is faithful as left (or right) B-module

(4) B is a commutative Noetherian semi-local ring without proper idempotent.

2. To begin with we will pick up some lemmas which we need to prove our main theorem. Some of them have already been known.

Lemma 1. Let A be a separable extension of B, and suppose that $A = B \oplus M$ with a B-B-submodule M such that $M^2 \subset B$. Set $M^2 = I$. Then we have $I^2 = I \neq 0$ and $IM = MI = M$.

Proof. See Theorem 1 and Proposition 2 [6].

The next lemma can be seen in [3] in more general form. But we will repeat here.

Lemma 2. Let R be a commutative ring with 1 and I a finitely generated idempotent ideal of R. Then we have $I = Re$ for some idempotent e of R.

Proof. Let $I = \Sigma_{i=1}^{m} Ra_i$. Then $a_i \in I = I^2 = \Sigma_{j=1}^{m} Ia_j$, and $a_i = \Sigma_{j=1}^{m} c_{ij}a_j$ for each i with $c_{ij} \in I$. Set $d = \det(E - (c_{ij}))$ $(= \det(\delta_{ij} - c_{ij}))$, where E is the unit matrix and δ_{ij} is the Kronecker's delta. Then, we see by direct computations that $d = 1 - e$ for some $e \in I$, and $da_j = 0$ for each j. Thus $(1 - e)I = 0$, in partiqular, $(1 - e)e = 0$. This means that $e^2 = e$, and $I = Re$.

The next lemma is also well known. So, we will state it without proof.

Lemma 3. If there exists B-B-submodules X and Y of A such that $XY = YX = B$, then X and Y are left as well as right B-progenerators, and there exist the following ring isomorphisms

$$i_X : B \longrightarrow [\mathrm{Hom}({}_BX, {}_BX)]^o, \qquad i_Y : B \longrightarrow [\mathrm{Hom}({}_BY, {}_BY)]^o$$

$$j_X : B \longrightarrow \mathrm{Hom}(X_B, X_B), \qquad j_Y : B \longrightarrow \mathrm{Hom}(Y_B, Y_B)$$

defined by $i_X(b)(x) = xb$, $j_X(b) = bx$ for each $b \in B$ and $x \in X$.

Lemma 4. Let B be a commutative subring of A (not necessarily contained in the center of A), and X and Y be the same as in Lemma 3. Then for each maximal ideal J of B, we have that $[X/JX: B/J] = [X/XJ: B/J] = 1$.

Proof. Set $n = [X/JX: B/J]$. Then, $\mathrm{Hom}({}_BX/JX, {}_BX/JX) = (B/J)_n$, the n×n-full matrix ring over B/J. On the other hand, we have by Lemma 3 that $B = [\mathrm{Hom}({}_BX, {}_BX)]^o$. But X is left B-projective. Hence there is a canonical ring homomorphism of $B = \mathrm{Hom}({}_BX, {}_BX)$ onto $\mathrm{Hom}({}_BX/JX, {}_BX/JX)$, where the former is commutative. Hence $(B/J)_n$ must be a commutative ring. This means that $n = 1$. Similarly, $[X/XJ: B/J] = 1$.

Remark. In the case where the left B-module structure of X coinsides with the right B-module structure of X, the obove lemma has already been shown in, for example, [1] Chap. 1 §5.

3. Now we will prove our main theorem.

First suppose condition (3). Set $M^2 = I$. Then by Lemma 1, $0 \neq I = I^2$, and I is finitely generated, since B is Noetherian. Hence, $I = Be$ for some $0 \neq e^2 = e \in B$, by Lemma 2. We have also $M = MI = Me$ by lemma 2. Then, $M(1 - e) = Me(1 - e) = 0$. But M is faithful as right B-module. Hence $e = 1$, and we see that $M^2 = B$. Then by Lemma 4, we see that $[M/JM; B/J] = 1$ for each maximal ideal J of B. Now let $\{J_1, J_2, \cdots, J_r\}$ be the set of all maximal ideals of B. Since MM = B, we see that $J_1 \cdots J_{i-1}J_{i+1} \cdots J_rM \not\subset J_iM$ for each i. Hence there exists $m_i \varepsilon J_1 \cdots J_{i-1}J_{i+1} \cdots J_rM$ such that $m_i \notin J_iM$. Set $m = \Sigma m_i$. Then $m \notin J_jM$ for each j. Therefore, $B/J_i(m + J_iM) = M/J_iM$, and $M = Bm + J_iM$ for each maximal ideal J_i of B. Then by Nakayama's Lemma we have M = Bm. Similarly, we have M = nB. Hence, $M^2 = nBm = B$. Thus there exists s in B such that nsm = 1. Then, n(smn - 1) = n - n = 0, and smn ε B. Therefore, M(smn - 1) = nB(smn - 1) = n(smn - 1)B = 0. But M is faithful as right B-module. Hence we have that 1 = (sm)n = (mn)s. Thus m, n and s are all units. Hence we can easily see that M = mB = Bm, and there exists an automorphism σ of B such that xm = $m\sigma(x)$ for all x in B. Set $u = m^2$ (ε $M^2 = B$). Then, u is a unit of B, and $xu = xmm = mm\sigma^2(x) = u\sigma^2(x)$ for each x in B. Furthermore, $um = m^3 = mu = m\sigma(u)$. This means that $\sigma(u) = u$. Now it is obvious that $A = B + Bm = B[X;\sigma]/(X^2 - u)$ by $m \longrightarrow X + (X^2 - u)$.

Next assume condition (1). Set $M^2 = I$, and let J be the radical of B. If $I \subset J$, we have IM = JM = M by Lemma 1. Then, M = 0, since M is left B-finitely generated. Thus $I \not\subset J$, which means that $M^2 = B$. On the othere hand, since B is a local ring, every finitely generated projective B-module is B-free of finite rank. This fact together with Lemma 3 shows that M = nB = Bm. Then, 1 = nsm for some s ε B. Since B has no proper idempotent, we see that smn = 1. Thus m, n and s are units. Then, for the same reason as obove, we see that A and B satisfy (T. P. R) with some σ. Assume condition (2), and suppose $I \subset J$. Then by Lemma 1, we have I = II = JI = I. But I is left B-finitely generated.

Hence I = 0, which contradicts to Lemma 1. Thus we have again that $M^2 = B$. Now we can follow the same lines as above. It is also easy to see that under condition (4) we can obtain the same conclusion by the same method.

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