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Spitzer's Markov chains and non-linear integral equations of the Hammerstein type

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Spitzer [5] has introduced Markov chains, whose space of "time parameters" is an infinite tree T, and whose state space is a set {-1, +1}. He investigates Gibbs distributions on T that are Markov chains of such construction. We generalize Spitzer's results to a case when the state space is a compact set. If the state space consists of two points as in a case of Spitzer, all Markov chains are reversible. So, in that case, the "time parameter" space T need not be equipped with a direction. But, since Markov chains may not be reversible in our case, we must introduce a direction into T.

Let T be an infinite directed tree, in which s branches emanate from every vertex and n branches flow into every vertex. Generalizing Spitzer's construction [5], we define Markov chains whose space of "time parameters" is the tree T, and whose state space is a compact metric measure space $(X,\mathcal{B},\mu)$. Let $F(x,y)$ be a measurable function on $X \times X$. We assume neither boundedness nor symmetry $F(x,y) = F(y,x)$ of F. If F satisfies

(A,1) $\int\int e^{-(n+s)F(x,y)}\mu(dx)\mu(dy) <+\infty$

or

(A,2) $\sup_x\{\int e^{-F(x,y)}\mu(dy), \int e^{-F(y,x)}\mu(dy)\} <+\infty,$

then we can define Gibbs distributions on T with the potential F ([1],[3]). Let $\mathcal{M}(F)$ be the set of Markov chains that are Gibbs distributions with the potential F.

Putting $X(x,M)=\{y\in X; F(x,y)\leqq M\}$ and $X^*(x,M)=\{y\in X; F(y,x)\leqq M\}$, we assume that there exist M and an integer k such that

$$\text{(A,3)}\quad \begin{cases} \mu^k\{(x_1,x_2,\cdots,x_k); \mu(X\setminus \bigcup_{i=1}^{k} X(x_i,M)) = 0\} > 0, \\ \mu^k\{(x_1,x_2,\cdots,x_k); \mu(X\setminus \bigcup_{i=1}^{k} X^*(x_i,M)) = 0\} > 0. \end{cases}$$

Theorem 1. Under the above assumptions, a Markov chain with the transition density p(x,y) belongs to $\mathcal{M}(F)$, if and only if p(x,y) has the expression;

$$p(x,y) = \lambda(s,n)u(x)^{-1}u(y)^s v(y)^{n-1}e^{-F(x,y)},$$

where $\lambda(s,n)$ is the Perron-Frobenius eigenvalue of the kernel $e^{-F(x,y)}$ if $s=n=1$, and $\lambda(s,n)=1$ if otherwise, and u and v are positive measurable functions satisfying the following integral equations of the Hammerstein type;

$$(*)\quad \begin{cases} u(x) = \lambda(s,n)\int e^{-F(x,y)}u(y)^s v(y)^{n-1}\mu(dy), \\ v(x) = \lambda(s,n)\int e^{-F(y,x)}u(y)^{s-1}v(y)^n\mu(dy), \\ \int u^s v^{n-1}d\mu = \int u^{s-1}v^n d\mu, \\ \int u d\mu = \int v d\mu = 1, \text{ if } s=n=1, \\ \int u^s v^n d\mu < +\infty. \end{cases}$$

The expression is unique. The invariant density h(x) of p(x,y) has the form

$$h(x) = c\, u(x)^s v(x)^n,$$

where c is a normalizing constant.

<u>Theorem 2</u>. The set $\mathcal{M}(F)$ is not empty, either if

(A,4) $\int e^{-F(x,y)}\mu(dy)$ and $\int e^{-F(y,x)}\mu(dy)$ do not depend on x,

or if

(A,5) $\sup_x\{\int e^{-(n+s)F(x,y)}\mu(dy), \int e^{-(n+s)F(y,x)}\mu(dy)\} <+\infty$

and

(A,6) $\sup_x\{\int e^{(n+s)(n+s-2)F(x,y)}\mu(dy), \int e^{(n+s)(n+s-2)F(y,x)}\mu(dy)\} <+\infty.$

We say that p(x,y) is <u>reversible</u>, if $h(x)p(x,y) = h(y)p(y,x)$. We say that a potential F is <u>uniformly symmetrizable</u>, if there exsists a symmetric potential $\hat{F}$ such that $\sup_{x,y}|F(x,y)-\hat{F}(x,y)| <+\infty$ and such that $\hat{F}$ determines the Hamiltonian which is equivalent to that determined by F.

<u>Theorem 3</u>. We assume (A,3), (A,5) and

(A,6)' $\sup_x\{\int e^{(n+s)(n+s-2)'F(x,y)}\mu(dy), \int e^{(n+s)(n+s-2)'F(y,x)}\mu(dy)\} <+\infty,$

where $(n+s)(n+s-2)' = \max\{(n+s)(n+s-2),1\}$. Then, the following three statements are equivalent to each other.

1) A potential F is uniformly symmetrizable.

2) There exists a reversible Markov chain in $\mathcal{M}(F)$.

3) All Markov chains in $\mathcal{M}(F)$ are reversible.

If F is symmetric, u and v in Theorem 1 can be regarded as positive eigenfunctions of the kernel $e^{-F(x,y)}u(y)^{s-1}v(y)^{n-1}$, which implies u=c v. Therefore,(*) reduces to

$$(**)\quad \begin{cases} u(x) = \lambda(s,n)\int e^{-F(x,y)}u(y)^{s+n-1}\mu(dy), \\ \int u\,d\mu = 1, \text{ if } s=n=1, \\ \int u^{s+n}d\mu <+\infty. \end{cases}$$

In case the state space is the unit circle S^1 which we identify with $[0,1)$, we can construct an example

$$u(x) = \int_0^1 \Gamma(x-y)u(y)^2 dy,$$

where Γ is positive, even and of C^∞-class and u is positive and non-constant. The Markov chain in $\mathcal{M}(-\log\Gamma)$ determined by u is not rotation-invariant. On the contrary, all Gibbs distributions in Z^2 with the state space S^1, whose potential is of finite range, of C^2-class and rotation-invariant, are also rotation--invariant [2].

In the following we consider potentials with the form βF, where $\beta > 0$ is called the <u>reciprocal temparature.</u>

<u>Theorem 4</u>. Assume (A,3) and

$$\text{(A,7)} \quad \sup_x\{\int e^{|F(x,y)|}\mu(dy),\ \int e^{|F(y,x)|}\mu(dy)\} < +\infty.$$

If β is sufficiently small, then $\mathcal{M}(\beta F)$ consists of a unique Markov chain.

<u>Theorem 5</u>. Let X be a finite set and let $\mu(\{i\}) > 0$ for all $i \in X$. Let F be a symmetric potential on X satisfying

$$\text{(A,8)} \qquad F(i,j) > F(j,j) + \frac{1}{n+s-1}|F(i,i)-F(j,j)|$$

for all $i \neq j \in X$. Then, the number of Markov chains in $\mathcal{M}(\beta F)$ is equal to $2^{\#X}-1$ for sufficiently large β, if $s+n > 2$.

Details are found in [4].

References

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