Dimension of product spaces

Yukihiro Kodama

Department of Mathematics, University of Tsukuba

For a topological space X, we denote by dim X the covering dimension of X in the sense of Katetov. Consider the following relations.

(I)   dim $(X \times Y) \leqq$ dim X + dim Y

(II)   dim $(X \times Y) >$ dim X

(III)   dim $(X \times Y)$ = dim X + dim Y

For the relation (I), B.A.Pasynkov proved the following beautiful theorem.

Theorem 1. [4] If the product space $X \times Y$ is rectangular, then the relation (I) holds.

This is a very strong theorem. In almost all cases where it was proved that the relation (I) held, the product $X \times Y$ was rectangular. Hence the following Yajima's theorem is interesting.

Theorem 2. [5] Let X be a collectionwise normal space which has a σ-closure-preserving closed cover by m-compact sets and Y be a subparacompact space with $\chi(Y) \leqq$ m. If $X \times Y$ is normal, then (I) holds.

Note that $X \times Y$ is not generally rectangular in Theorem 2.

Anderson and Keisler [1] constructed a separable metric

space X_n, $n = 1,2,\ldots$, with the following property: $\dim X_n^k = n$ for $k = 1,2,\ldots$, ∞ , where X_n^k is the k-fold product of X_n. Thus the following problem is interesting.

Problem 1. For a paracompact space X and a compact space Y with $\dim Y > 0$, does the relation (II) hold ?

A partial answer to Problem 1 holds.

Theorem 3 [3] If X is paracompact and Y is compact such that $\dim Y > 0$ and the 1-dimensional Čech cohomology group $\check{H}^1(Y) = 0$, then (II) holds.

Corollary. If Y is a non-degenerate ASR, that is, a compact space with trivial shape, then (II) holds for every paracompact space X.

Problem 2. (K.Borsuk) If Y is a compact ANR, then does (III) hold for every paracompact space X ?

It is known [2] that if Y is a 2-dimensional compact ANR, then (III) holds for every paracompact space X.

Problem 3. If Y is a continuous curve, then does (III) hold for every paracompact space X.

The following partial answer to Problem 3 is known.

Theorem 4 [3] If Y is a movable curve, then (III) holds for every paracompact space X.

References

[1] R.D.Anderson and J.E.Keisler, An example in dimension theory, Proc.Amer.Math.Soc.,18(1967),709-713.

[2] Y.Kodama, Cohomological dimension theory, Appendix:Dimension theory, K.Nagami, Academic Press, 1970.

[3] Y.Kodama, A remark on the dimensional fullvaluedness, Proc. Japan Acad.,53(1977), 113-115.

[4]   B.A.Pasynkov, Factorization theorems on dimension theory,
      YCΠEX MAT.NAUK, 1981:36:3(219), 147-175.

[5]   Y.Yajima, Topological games and products II, to appear in
      Fund.Math.