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The Satisfiability Problems for Some Classes of Extended Horn Sets in the Propositional Logic

Susumu Yamasaki and Shuji Doshita
Department of Information Science, Kyoto University

1. Introduction

The Horn set of clauses has important features in the logic of programs and the theory of computing. First, the Horn set is a program in the programming language PROLOG proposed by R.Kowalski, the computation of which is regarded as unit or input resolution deduction and can simulate that of Turing machines [10,13]. Secondly, the satisfiability or the unsatisfiability of any Horn set in the propositional logic can be decided by unit or input resolution in deterministic polynomial time. Also the problem of determining whether there is a unit resolution refutation from a given propositional set of clauses is complete for deterministic polynomial time [7].

Unit and input resolutions were proposed by C.L.Chang [2]. They are complete inference rules to detect unsatisfiability of the Horn set [6]. In the propositional logic, they are complete in detecting the satisfiability of Horn sets. Unit resolution in the propositional logic is an inference rule to derive a formula (called a resolvent) from a literal and a formula (a clause) containing its complementary literal. Unit resolution is equivalent to input resolution, whose deduction takes a linear form in which center clauses are resolvents and side clauses are given input clauses. In an input deduction, the center clauses (resolvents) cannot be side clauses [2]. Thus the input resolution deduction is a linear deduction with such strong restrictions.

In this manuscript, we formulate a Restricted Linear (RL) deduction in which some center clauses can be used as side clauses, by relaxing the restriction in the input resolution deduction. From the constructive viewpoint, we show that the RL deduction is formed by layering linear deductions each of which corresponds to an input resolution refutation. Then we call this type of RL deduction by another name: a Linear Layered Resolution deduction based on Input resolutions (an LLRI deduction). Next, as an extension of LLRI, we formulate a nonlinear form of layered resolution deductions based on input resolutions (an NLRI deduction) in which resolvents are generated by the LLRI deduction from both input clauses and unit clauses obtained by the LLRI deduction.

On the basis of the LLRI deduction and the NLRI deduction, we characterize some classes of extended Horn sets in the propositional logic which are more general than the class of Horn sets and for which the satisfiability problems can be solved by LLRI and NLRI deductions in deterministic polynomial time and complete problems for deterministic polynomial time.

2. Resolution and Horn set

2.1 Set of Clauses

A propositional formula in clausal form is mentioned in this manuscript. A formula in clausal form is a set of clauses. A clause is a set of literals. When the clause is empty, it is called the empty clause and is denoted by $\Box$. The clause consisting of only one literal is called the unit clause. A literal is either a positive literal +P or a negative literal -P, where P is an atom denoted by the predicate symbol P.

A valuation v is a function from the set of atoms to $\{0,1\}$. The value $\bar{v}$ (relative to v) of literals, clauses and sets of clauses are defined as follows: $\bar{v}(+P)=v(P)$. $\bar{v}(-P)=1-v(P)$. $\bar{v}(\{L_1,L_2,\ldots,L_m\})=\max_i \bar{v}(L_i)$, where L_i for $1\leq i\leq m$ is a literal. $\bar{v}(\Box)=0$. $\bar{v}(\{C_1,C_2,\ldots,C_n\})=\min_j \bar{v}(C_j)$, where C_j for $1\leq j\leq n$ is a clause. A formula in clausal form is satisfiable if there is some valuation for which the value of the formula is 1, and unsatisfiable otherwise [5,16].

2.2 Resolution

A resolution is an inference rule to derive a clause $C_1 \cup C_2$ from two clauses $C_1 \cup \{+P\}$ and $C_2 \cup \{-P\}$, where C_1 and C_2 denote the sets of literals, and P denotes an atom. The derived clause $C_1 \cup C_2$ is denoted by $Res(C_1 \cup \{+P\}, C_2 \cup \{-P\})$, and is called a resolvent of parent clauses $C_1 \cup \{+P\}$ and $C_2 \cup \{-P\}$. In this case +P and -P are called the literals resolved upon.

The resolution in the propositional logic is a powerful inference rule to decide the unsatisfiability or satisfiability of sets of clauses.

<u>Definition</u> 1 [2]: Given a set S of clauses, a (resolution) deduction of C from S is a finite sequence $C_1,C_2,\ldots,C_k$ of clauses such that each C_i is either a clause in S or a resolvent of clauses preceding C_i, and $C_k=C$. A deduction of the empty clause from S is called a (resolution) refutation from S.

<u>Definition</u> 2 [2]: Unit resolution is a resolution in which a resolvent is obtained by using at least one unit clause. A unit deduction is a deduction in which every resolution is a unit resolution.

<u>Definition</u> 3 [2]: Input resolution is a resolution in which one of the parent clauses is a given input clause. An input deduction is a deduction in which every resolution is an input resolution.

2.3 Horn Set

A Horn clause is a clause which contains at most one positive literal. A Horn set is a set of Horn clauses.

Theorem 1 [6]: If S is an unsatisfiable Horn set, then there is an input (unit) refutation from S.

Definition 4 [6]: A set S of clauses is minimally unsatisfiable if S is unsatisfiable and no proper subset of S is unsatisfiable.

Theorem 2 [6]: Let S be a minimally unsatisfiable, and input refutable set, then there is a set S' such that S' is a renaming of S and S' is a Horn set.

3. Layered Resolutions Based on Input Resolutions

Definition 5: Given a set S of clauses and a clause C_0 in S, an RL (Restricted Linear) deduction of C_n with top clause C_0 is a deduction of the linear form of resolution in which:

1. For i=0,1,...,n-1, C_{i+1} is a resolvent of C_i (called a center clause), and B_{i+1} (called a side clause), and each B_{i+1} is either in S, or in the set MC as defined below. C_n is in MC.
2. (the set MC)

(1) (i) C_0 is in MC. (ii) If C_i is in MC and j (greater than i) is the least integer under the following condition, then C_j is also in MC. In this case C_j is said to be adjacent to C_i.

(2) (condition)

(i) $C_i \supsetneq C_j$. (ii) The literals resolved upon in $C_i, C_{i+1}, \ldots, C_{j-1}$ cannot be contained in C_j.

An RL deduction of the empty clause is called an RL refutation.

Assume that there is an RL deduction in which C_j is adjacent to C_i and $C_k=Res(C_{k-1},B_k')$ for $i+1 \le k \le j$. Let $C_k'=C_k$ if C_k is a unit clause, $C_k'=C_k-C_j$ otherwise, for $i \le k \le j$. Let $B_k'=B_k$ if B_k is a unit clause, $B_k'=B_k-C_j$ otherwise, for $i \le k \le j$. Then $C_k'=Res(C_{k-1},B_k')$ for $i \le k \le j$, and center clauses C_k' and side clauses B_k' for $i \le k \le j$ form an input refutation with top clause C_i'.

Thus, an RL deduction is formed by layering linear deductions each of which corresponds to an input refutation. Therefore we call an RL deduction by another name: a Linear Layered Resolution deduction based on Input resolutions (an LLRI deduction).

Lemma 3: Assume that an RL (LLRI) deduction of C_j exists and C_j is adjacent to C_i. Then there is an RL deduction of C_j with top clause C_i in which any clause in MC except C_i is not used as a side clause.

Definition 6: Let C_1 and C_2 be clauses such that $C_1=C_2 \cup \{L\}$ for some literal L. Then C_2 is said to have a relation Re with C_1.

Lemma 4: Assume that there is an RL (LLRI) deduction of C_j with top clause C_i. Then there is an RL deduction of C_j with top clause C_i such that any clause C in MC such that $C_i \supset C \supset C_j$ has a relation Re with its adjacent clause.

It is concluded by the lemma that there is some input refutation with the specified unit clause as top clause for each RL deduction of a clause with its adjacent one as top clause.

Definition 7: A clause C in S is said to be admissible to C' with respect to S if there is a unit deduction of C' from S, $C_1, C_2, \ldots, C_k$ such that C_1=C and C_k=C'.
Lemma 5: Let S be a set of clauses, and assume that there is a unit refutation from S. Then there is an input refutation from S with any clause C_0, admissible to the empty clause with respect to S, as top clause.
Lemma 6: Let S be a set of clauses. If there is an input refutation with top clause C_0, then C_0 is admissible to the empty clause with respect to S.

By Lemmas 5 and 6, input refutability from a given set of clauses with a unit clause as top clause can be decided by the algorithm to determine whether there is a unit refutation from the set in which the unit clause is admissible to $\Box$ with respect to the set. The algorithm can be constructed by gathering and storing the unit clauses which are derived from the specified unit clause by unit resolution and contribute to the derivation of the empty clause.
Lemma 7: There is a deterministic polinomial time algorithm to determine whether there is an input refutation with the specified unit clause as top clause from a given set of clauses.

By Lemmas 4 and 7, we can construct a bottom-up method to detect an RL refutation with some input clause as top clause by using unit (input) resolution repeatedly, and obtain the theorem.
Theorem 8: The problem of determining whether there is an RL refutation from a given set of clauses can be solved in deterministic polynomial time.

Next we formulate a nonlinear (form of) deduction by extending the LLRI deduction in the way that clauses which consist of a limited number of literals and are obtained by LLRI deductions can be used as side clauses of other LLRI deductions.
Definition 8: We define a Nonlinear (form of) Layered Resolution based on Input resolutions of type k (k-NLRI) as follows:
If C is a clause containing not greater than k literals, an RL (LLRI) deduction of C from $\{C_1, C_2, \ldots, C_m\}$ is denoted by k-LLRI($C_1, C_2, \ldots, C_m$:C).
A k-NLRI deduction of C from S=$\{C_1, C_2, \ldots, C_m\}$ for C containing not greater than k

literals is denoted by k-NLRI($C_1, C_2, \ldots, C_m$:C) and is defined recursively.

1. k-LLRI($B_1, B_2, \ldots, B_n$:C) is also k-NLRI($B_1, B_2, \ldots, B_n$:C), where each T_i is in S.
2. If there are k-LLRI($B_1, B_2, \ldots, B_k$:C_1) for each T_i and a unit clause C_1 in S, and k-LLRI($A_1, A_2, \ldots, A_j, C_1$:C) for each A_i in S, then there is k-NLRI($B_1, B_2, \ldots, B_k, A_1, A_2, \ldots, A_j$:C).
3. The deduction defined by 1 and 2 recursively is a k-NLRI deduction.
4. All k-NLRI deductions are defined by applying the above rules.

We can construct an algorithm to decide an NLRI refutability from a set of clauses in two steps.

1. For a clause containing not greater than k literals, decide whether there is an LLRI deduction of the clause. If such a deduction exists, add it to an input set.
2. Decide whether there is an LLRI refutation from the input set and the added clauses.

Theorem 9: The problem of determining whether there is a k-NLRI refutation from a given set of clauses can be solved in deterministic polynomial time.

4. Satisfiability Problems for Some Classes of Extended Horn Sets

Definition 9: A clause C_1 is said to be a C-Horn clause if $C_1 = C \cup C_2$ for some set C of positive literals and nonempty Horn clause C_2 such that $C \cap C_2 = \phi$.

Definition 10: We define a class $\mathfrak{S}_1$ to be $\{S \mid S = \bigcup_{i=1}^{n} \{C_i\text{-Horn clause}\}$, for some $C_1, C_2, \ldots, C_n$ such that $C_1 \supset C_2 \supset \ldots \supset C_n\}$. We call a set in the class an extended Horn set in $\mathfrak{S}_1$.

We can obtain the followin theorem.

Theorem 10: Let S be a set in $\mathfrak{S}_1$. Then S is unsatisfiable if and only if there is an LLRI refutation from S.

Definition 11: A set S of clauses is said to be minimally LLRI refutable if there is an LLRI refutation from S and there is no LLRI refutation from any proper subset of S.

Theorem 11: If S is a minimally LLRI refutable set of clauses , then there is a set S_1 as follows;

1. S_1 is an unsatisfiable extended Horn set in $\mathfrak{S}_1$.
2. S is a logical consequence of $S_1 \cup S_0$ for some set S_0 of clauses each of which contains just two literals.

We can construct a deterministic polynomial time algorithm to determine whether a given set S of clauses is in the class $\mathfrak{S}_1$.

It is concluded from [6,7] that the unsatisfiability problem or the satisfiability problem for the Horn sets in the propositional logic is P-complete.

Let D_1 and D_2 be domains of sets of clauses. Let RED1:$D_1 \rightarrow D_2$ be a function defined as follows: For a set $S=\{C_1,C_2,\ldots,C_m\}$ in D_1, RED1(S)$=\{C_1 \cup \{-P\}, C_2 \cup \{-P\}, \ldots, C_m \cup \{-P\}, \{P,P_1,P_2\}, \{-P_1,P_2\}, \{P_1,-P_2\}, \{-P_1,-P_2\}\}$, where P, P_1 and P_2 are atoms not appearing in S.

Then (1) RED1 is log(•)-space computable, (2) RED1(S) is in $\mathfrak{S}_1$ if and only if S is a Horn set, and (3) S is unsatisfiable if and only if RED1(S) is unsatisfiable. Thus we can obtain the theorem.

Theorem 12: The unsatisfiability problem or the satisfiability problem for the extended Horn sets in $\mathfrak{S}_1$ is P-complete.

Definition 12: We define a binary relation on Re on the class $\mathfrak{S}_1$ as follows: $S_i \underline{Re} S_j$ for S_i and S_j in $\mathfrak{S}_1-\{\text{Horn sets}\}$ if and only if any clause in S_i contains no common atom with any clause in S_j except that at most one negative literal $-P_j$ may appear in S_i and at most one negative literal $-P_i$ may appear in S_j, where $+P_i$ is a positive literal contained in each clause in S_i and $+P_j$ is a positive literal contained in each clause in S_j.

The sets S_i in $\mathfrak{S}_1-\{\text{Horn sets}\}$ for $1 \leq i \leq n$ are said to satisfy the condition $\mathfrak{C}$ if $S_i \underline{Re} S_j$ for any i and j $(1 \leq i \neq j \leq n)$.

Definition 13: We define a class $\mathfrak{S}_2$ to be $\{S \mid S= S_0 \cup \bigcup_{i=1}^{m} S_i$ for some Horn set S_0 and some $S_1, S_2, \ldots, S_m$ in $\mathfrak{S}_1-\{\text{Horn sets}\}$ satisfying the condition $\mathfrak{C}\}$. A set in $\mathfrak{S}_2$ is called an extended Horn sets in $\mathfrak{S}_2$.

We can obtain the following theorems.

Theorem 13: Let S be an unsatisfiable extended Horn set in $\mathfrak{S}_2$. Then S is unsatisfiable if and only if there is a 1-NLRI refutation from S.

Definition 14: A set S of clauses is said to be minimally 1-NLRI refutable if there is a 1-NLRI refutation from S and there is no 1-NLRI refutation from any proper subset of S.

Theorem 14: If S is minimally 1-NLRI refutable, then there is a set S_1 in $\mathfrak{S}_2$ such that

1. S_1 is unsatisfiable.
2. S_1 is a logical consequence of $S \cup S_0$ for some set S_0 of clauses each of which contains just two literals.

A deterministic polynomial time algorithm to determine whether a given set S is in the class $\mathfrak{S}_2$ can be constructed.

Now let D_2 and D_3 be domains of sets of clauses and a function RED2:$D_2 \rightarrow D_3$ be defied as follows: For a set $S=\{C_1,C_2,\ldots,C_m\}$ in $\mathcal{S}_1$, $RED2(S)=\{C_1\cup\{+P\},C_2\cup\{+P\},\ldots,C_m\cup\{+P\},\{-P,+P_1,+P_2,+P_3\},\{-P_2,+P_3\},\{+P_2,-P_3\},\{-P_2,-P_3\},\{-P,-P_1,+P_4,+P_5\},\{-P_4,+P_5\},\{+P_4,-P_5\},\{-P_4,-P_5\}\}$, where P,P_1,P_2,P_3,P_4 and P_5 are atoms not appearing in S. Let $S_1=\{C_1\cup\{+P\},C_2\cup\{+P\},\ldots,C_m\cup\{+P\}\}$, $S_2=\{-P,+P_1,+P_2,+P_3\}$ and $S_3=\{-P,-P_1,+P_4,+P_5\}$. Then $RED2(S)=S_1\cup S_2\cup S_3\cup\{\{-P_4,+P_5\},\{+P_4,-P_5\},\{-P_4,+P_5\}\}$. RED2(S) is in $\mathcal{S}_2$ only if S is in $\mathcal{S}_1$. If S is in $\mathcal{S}_1$, then S_1 is in $\mathcal{S}_1$, and $S_i \underline{Re} S_j$ for $1\leq i\neq j\leq 3$, and so RED2(S) is in $\mathcal{S}_2$. Therefore (1) RED2(S) is in $\mathcal{S}_2$ if and only if S is in $\mathcal{S}_1$. Also (2) RED2 is log(•)-space computable, and (3) S is unsatisfiable if and only if RED2(S) is unsatisfiable.

<u>Theorem</u> 15: The unsatisfiability problem or the satisfiability problem for the extended Horn sets in $\mathcal{S}_2$ is P-complete.

5. Concluding Remarks

In this manuscript we characterized some classes of extended Horn sets for which the satisfiability problems are P-complete, on the basis of proposed layered resolutions based on input resolutions. We will investigate a class of extended Horn sets characterized by k-NLRI deductions for which the satisfiability problem is P-complete.

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