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On a 3-valued logic connected with incomplete information data bases

by

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§0 Introduction

In [1], Lipski proposed a mathematical model of data bases with incomplete information and discussed some problems related to it. According to his proposal, propositions which express queries to an information storage and retrieval system can be regarded as a special kind of formulas of the first-order predicate logic. So, in [2] he gave two ways (i.e., *external* and *internal*) of interpreting formulas of the predicate logic, by making use of models of data bases with incomplete information. In regard to this interpretation, some similarities to Kripke models for modal logic are known. In fact, some relationships to modal logic S4 were mentioned in [2].

In [3], we introduce a second-order predicate logic corresponding to this query language and solved some interesting problems about the decidability of this language. In this paper, we propose a 3-valued ($\{1, 1/2, 0\}$) logic based on this model instead of the above second-order predicate logic. That is, we give a translation of formulas of the query language, called *extended formulas*, into formulas of this 3-valued logic and show that an extended formula is true in every interpretation iff the corresponding formula of this 3-valued logic is valid. This theorem is proved by making use of the following ideas:

(1) Incomplete information corresponds to the value 1/2 of the 3-valued logic.

(2) Formula of modal logic can be induced into the first-order predicate logic by introducing a new sort of domain.

§1 Preliminaries

In this paper, we will use almost the same terminology and notations as in [2]. First, we will give a brief account of *internal interpretations* for a query language.

A first-order language L is a language which consists of a list of countable n-ary predicate symbols $P^n, Q^n, \ldots, P^n_1, P^n_2, \ldots$, for each $n \geq 1$, a list of countable individual variables $x, y, \ldots, x_1, x_2, \ldots$, the logical connectives $\neg$, $\wedge$, and the quantifier $\forall$. Other connectives $\vee$, $\supset$, $\equiv$, $\exists$ can be defined as abbreviations in the usual way. We suppose that L does not contain any function symbol and any individual or predicate constants. P^n is sometimes denoted by P. *First-order formulas* of L are defined in the usual way. Next, we will add a unary logical connective $\Box$ to L. The language thus obtained is denoted by L*. (First-order) formulas of L* are called *extended formulas*. In the following, formulas will be denoted by $\phi, \psi, \ldots$, or $\phi(x_1, \ldots, x_n)$, $\psi(x_1, \ldots, x_n), \ldots$. (Some of variables $x_1, \ldots, x_n$ may not occur in $\phi(x_1, \ldots, x_n)$ and other variables may occur in it.)

Following Lipski [2], we will introduce internal interpretations of extended formulas.

Definition 1.1 An *incomplete model* (or a *model* for short) is a triple $M=\langle X,u,U\rangle$, where X is a nonempty set called the individual domain of M, and u and U are mappings which associate some subsets $u(P) \subseteq U(P) \subseteq X^n$ for every n-ary predicate symbol P ($n>1$).

If $u=U$ holds in a model $M=\langle X,u,U\rangle$, then M is said to be *complete*. Complete models are nothing but ordinary models for the first-order formulas, as explained later.

Definition 1.2 Given two models $M_1=\langle X,u_1,U_1\rangle$ and $M_2=\langle X,u_2,U_2\rangle$ with the same individual domain X, M_2 is an *extension* of M_1 ($M_1 \preccurlyeq M_2$ or $M_2 \succcurlyeq M_1$, in symbol) if and only if for every predicate symbol P $u_1(P) \subseteq u_2(P) \subseteq U_2(P) \subseteq U_1(P)$.

Let $\phi(x_1,\ldots,x_n)$ be any extended formula with free individual variables $x_1,\ldots,x_n$. For any model $M=\langle X,u,U\rangle$ and $a_1,\ldots,a_n \in X$, we want to define the notation "*$\phi(x_1,\ldots,x_n)$ is satisfied in M when $x_1,\ldots x_n$ are interpreted as $a_1,\ldots,a_n$, respectively*", in symbol

$$M \models \phi(a_1,\ldots,a_n).$$

To do so, we first extend our language L^* by adding a new individual constant $\bar{a}$ for each $a \in X$. (By abuse of symbol, we will use the same letter a for $\bar{a}$, in the following.) The language thus obtained is denoted by $L^*[M]$.

Definition 1.3 Let $M=\langle X,u,U\rangle$ be any model. For each closed extended formula ϕ of $L^*[M]$, define $M \models \phi$ recursively as follows:

1) $M \models P(a_1,\ldots,a_n)$ iff $(a_1,\ldots,a_n) \in u(P)$,

where P is an n-ary predicate symbol,

2) $M \models \neg\psi$ iff not $M \models \psi$,

3) $M \models \psi \wedge \theta$ iff $M \models \psi$ and $M \models \theta$,

4) $M \models \forall x \psi(x)$ iff for every $a \in X$ $M \models \psi(a)$,

5) $M \models \Box\psi$ iff for every $M' \geqslant M$ $M' \models \psi$.

Next, let $\phi(x_1,\ldots,x_n)$ be any extended formula of L* with free individual variables $x_1,\ldots,x_n$. Then, define

$$M \models \phi(x_1,\ldots,x_n) \quad \text{iff} \quad M \models \forall x_1 \ldots \forall x_n \phi(x_1,\ldots,x_n).$$

Notice that $\forall x_1 \ldots \forall x_n \phi(x_1,\ldots,x_n)$ is a closed formula of L*[M], in the above definition. By making use of the notation in [2], we can express the relation $\models$ as

$$M \models \phi(a_1,\ldots,a_n) \quad \text{iff} \quad (a_1,\ldots,a_n) \in \|\phi(x_1,\ldots,x_n)\|_M$$

for any extended formula $\phi(x_1,\ldots,x_n)$ with free individual variables $x_1,\ldots,x_n$. When M is a complete model, the definition of $M \models \phi$ coincides with the ordinary one, for every first-order formula ϕ .

Definition 1.4 Let ϕ and ψ be arbitrary extended formulas.

1) ϕ is *internally valid* if and only if $M \models \phi$ holds for every model M,

2) ϕ and ψ are *internally equivalent* if and only if $\phi \equiv \psi$ is internally valid.

§2 3-valued predicate logic ${}_3L$

We consider a 3-valued predicate logic ${}_3L$. The symbols of this logic are the same as in the usual predicate logic except the logical symbols and the special individual symbols. Further, well-defined formulas (wff's) are defined in the following way:

(1) The arguments of every n-ary predicate symbol P(.,...,.) must be occupied by the special individual symbol w_i ($0 \leq i$) in the n-th argument.

(2) Other construction rules except (1) are the same as in the usual one.

Logical symbols are $\wedge, \vee, \supset, \neg, J_1, J_{1/2}, J_0$. $\wedge, \vee$, and $\supset$ are duadic and $J_1, J_{1/2}, J_0, \neg$ are monadic.

The semantics of this logic is defined as follows:

First, the truth values are 1,1/2,0 and they mean *truth*, *unknown*, *false*, respectively. Domains of individual constants have two sorts X and W. X is the usual domain and W is the special domain, called *world*. The ordinary individual variable ranges over X, but the special individual variable w_i over W. Let $\mathfrak{A}$ be a wff of ${}_3L$. Then, $v(\mathfrak{A})$ stands for a valuation v of $\mathfrak{A}$. The truth value function of $\vee, \wedge, \supset, \neg, J_1, J_{1/2}, J_0$ are defined as follows:

$v(\mathfrak{A} \vee \mathfrak{B}) = \max(v(\mathfrak{A}), v(\mathfrak{B}))$,

$v(\mathfrak{A} \wedge \mathfrak{B}) = \min(v(\mathfrak{A}), v(\mathfrak{B}))$,

$v(\mathfrak{A} \supset \mathfrak{B}) = \min(1, v(\mathfrak{B}) - v(\mathfrak{A}) + 1)$,

$v(\neg \mathfrak{A}) = 1 - v(\mathfrak{A})$,

$$v(J_i \mathfrak{A}) = \begin{cases} 1 & \text{if } v(\mathfrak{A}) = i \\ 0 & \text{otherwise.} \end{cases}$$

Further, we use in the usual way the following definition for closed wff's $\forall x \mathfrak{A}$ and $\forall w_i \mathfrak{A}$. That is as follows:

$v(\forall x \mathfrak{A}(x)) = 1$ if x is an ordinary individual variable and for all $a \in X$ $v(\mathfrak{A}(a)) = 1$,

$= 1/2$ if x is an ordinary individual variable and for all $a \in X$ $v(\mathfrak{A}(a)) \geq 1/2$ and there exists $b \in X$ such that $v(\mathfrak{A}(b)) = 1/2$,

$= 0$ if x is an ordinary individual variable and there exists $a \in X$ such that $v(\mathfrak{A}(a)) = 0$.

$v(\forall w_i \mathfrak{A}(w_i)) = 1$ if w_i is a special individual variable and for all $r \in W$, $v(\mathfrak{A}(r)) = 1$,

$= 1/2$ if w_i is a special individual variable and for all $r \in W$ $v(\mathfrak{A}(r)) \geq 1/2$ and there exists $s \in W$ such that $v(\mathfrak{A}(s)) = 1/2$,

$= 0$ if w_i is a special individual variable and there exists $r \in W$ such that $v(\mathfrak{A}(r)) = 0$.

$\exists x \mathfrak{A}$ and $\exists w_i \mathfrak{A}$ are defined in the similar way.

The validity and satisfiability of wff $\mathfrak{A}$ of ${}_3L$ are also defined in the usual way.

§3 Theorems

Let S_1 and S_2 be the set of extended wff's and the set of wff's of ${}_3L$, respectively. Also, define a transformation τ on the set S_2 as follows:

$\tau(\mathfrak{A})$ is a formula obtained from $\mathfrak{A}$ by replacing (irrespective of *free* or *bounded*) $P(x_1,\ldots,x_n,w_i)$ by $P(x_1,\ldots,x_n,w_{i+1})$ for each P and each i of $\mathfrak{A}$.

Then, we define a mapping $f: S_1 \to S_2$ as follows:

(1) If $\phi = P(x_1,\ldots,x_n)$ then $f(\phi) = J_1P(x_1,\ldots,x_n,w_0)$,

(2) If $\phi = \neg\psi$ then $f(\phi) = \neg f(\psi)$,

(3) If $\phi = \psi \wedge \theta$ then $f(\phi) = f(\psi) \wedge f(\theta)$,

(4) If $\phi = \forall x\psi$ then $f(\phi) = \forall x f(\psi)$,

(5) Let $\phi = \Box\psi$ and $P_1,\ldots,P_k$ be all predicate symbols appearing in ϕ. Then,

$$f(\phi) = \forall w_1(((J_1\mathbf{P}(x_1,\ldots,x_n,w_0) \preccurlyeq J_1\mathbf{P}(x_1,\ldots,x_n,w_1)) \wedge (J_1\mathbf{P}(x_1,\ldots,x_n,w_1) \vee J_{1/2}\mathbf{P}(x_1,\ldots,x_n,w_1) \preccurlyeq J_1\mathbf{P}(x_1,\ldots,x_n,w_0) \vee J_{1/2}\mathbf{P}(x_1,\ldots,x_n,w_0)) \supset \tau(f(\psi))),$$

where $J_1\mathbf{P}(x_1,\ldots,x_n,w_0) \preccurlyeq J_1\mathbf{P}(x_1,\ldots,x_n,w_1)$ means $(J_1P_1(x_1,\ldots,x_n,w_0) \preccurlyeq J_1P_1(x_1,\ldots,x_n,w_1)) \wedge \cdots \wedge (J_1P_k(x_1,\ldots,x_n,w_0) \preccurlyeq J_1P_k(x_1,\ldots,x_n,w_1))$ and $J_1\mathbf{P}(x_1,\ldots,x_n,w_1) \vee J_{1/2}\mathbf{P}(x_1,\ldots,x_n,w_1) \preccurlyeq J_1\mathbf{P}(x_1,\ldots,x_n,w_0) \vee J_{1/2}\mathbf{P}(x_1,\ldots,x_n,w_0)$ has the similar meaning.

Lemma 3.1 Let $M=\langle X,u,U\rangle$ and ${}_3M=\langle X\cup W,v,v\rangle$ be models for L^* and ${}_3L$, respectively. Let $\bar{q}_m$ be a constant such that the corresponding q_m is in W. Moreover, suppose that for every predicate symbol P of L^* $u(P)=v(J_1P(\ldots,q_m))$ and $U(P)= v(J_1P(\ldots,q_m) \vee J_{1/2}P(\ldots,q_m))$ hold. Then, for any closed extended formula ϕ of $L^*[M]$

$$M \models \phi \quad \text{iff} \quad {}_3M \models \tau^m(f(\phi(q_0)))),$$

where $\tau^m(f(\phi(q_0)))$ means a formula obtained from $\tau^m(f(\phi))$ by substituting q_m for a special individual variable w_m.

Proof

We will show this lemma for every M and ${}_3M$ and m, by the induction on number of logical connectives in ϕ.

(1) The case where ϕ is $P(a_1,\ldots,a_n)$ for some $a_1,\ldots,a_n \in X$.

Then, $\tau^m(f(\phi(q_0)))=J_1(a_1,\ldots,a_n,q_m)$.

So, $M \models P(a_1,\ldots,a_n)$ iff $(a_1,\ldots,a_n) \in u(P)=v(J_1P(a_1,\ldots,a_n,q_m))$

iff ${}_3M \models \tau^m(f(P(a_1,\ldots,a_n,q_0)))$.

(2) Induction step

We will prove this lemma only for the case where ϕ is of the form $\Box\psi$. Other cases can be provable easily. For the sake of brevity, we suppose that predicate symbols appearing in ϕ is only P and that P is k-ary.

By the definition,

$M \models \phi$ iff for every $M' \geqslant M$ $M' \models \psi$.

So, it is sufficient to show that

(3.1) for every $M' \geqslant M$ $M' \models \psi$

if and only if

(3.2) ${}_3M \models \tau^m(f(\phi(q_0)))$.

We remark here that

$$\begin{aligned}\tau^m(f(\phi(q_0))) = \forall w_{m+1}(&(J_1P(x_1,\ldots,x_k,q_m) \leqslant J_1P(x_1,\ldots,x_k,w_{m+1}))\\ &\wedge(J_1P(x_1,\ldots,x_k,w_{m+1}) \vee J_{1/2}P(x_1,\ldots,x_k,w_{m+1})\\ &\leqslant J_1P(x_1,\ldots,x_k,q_m) \vee J_{1/2}P(x_1,\ldots,x_k,q_m))\\ &\supset \tau^{m+1}(f(\psi(q_0)))).\end{aligned}$$

We consider $J_1P(x_1,\ldots,x_k,q_m)$, $J_1P(x_1,\ldots,x_k,q_m) \vee J_{1/2}P(x,\ldots,x_k,q_m)$ as the corresponding formulas of P^m_*, P^{m*} in [3], respectively.

First, let us assume that (3.1) holds. For an arbitrary constant q_{m+1}, let $A=J_1P(x_1,\ldots,x_k, q_{m+1})$ and $B=J_1P(x_1,\ldots,x_k, q_{m+1}) \vee J_{1/2}P(x_1,\ldots,x_k,q_{m+1})$ such that

(3.3)

$_3M \models J_1P(x_1,\ldots,x_k,q_m) \leqslant A \leqslant B$

$\leqslant J_1P(x_1,\ldots,x_k,q_m) \vee J_{1/2}P(x_1,\ldots,x_k,q_m).$

Now, let us define a model $M'=\langle X,u',U'\rangle$ by $u'(P)=v(A)$ and $U'(P)=v(B)$. Since $M \leqslant M'$ holds, $M' \models \psi$. By the hypothesis of induction, we have

(3.4) $\quad {}_3M \models (\tau^{m+1}(f(\psi(q_0))))^+,$

where $(\tau^{m+1}(f(\psi(q_0))))^+$ denotes the formula obtained from $\tau^{m+1}(f(\psi(q_0)))$ by replacing each occurence of $J_1P(x_1,\ldots,x_k,w_{m+1})$ and $J_1P(x_1,\ldots,x_k,w_{m+1}) \vee J_{1/2}P(x_1,\ldots,x_k,w_{m+1})$ by predicate constants A and B, respectively. Since (3.4) holds from (3.3) for every A and B, (3.2) holds.

Conversely, suppose that (3.2) holds. Let $M'=\langle X,u',U'\rangle$ be any model such that $M \leqslant M'$. Define subsets A and B of X^k by $u'(P)=A$ and $U'(P)=B$, respectively. In this case, by making use of constant q_{m+1} in W and v of $_3M$, A and B are always representable by $\{(x_1,\ldots,x_k) \mid v(J_1P(x_1,\ldots,x_k,q_{m+1}))\}$ and $\{(x_1,\ldots,x_k) \mid v(J_1P(x_1,\ldots,x_k,q_{m+1}) \vee J_{1/2}P(x_1,\ldots,x_k,q_{m+1}))\}$, respectively.

Then, it holds that for the above $_3M$

$_3M \models J_1P(x_1,\ldots,x_k,q_m) \leqslant A \leqslant B$

$\leqslant J_1P(x_1,\ldots,x_k,q_m) \vee J_{1/2}P(x_1,\ldots,x_k,q_m)$

because $M \leqslant M'$. So, $_3M \models (\tau^{m+1}(f(\psi(q_0))))^+$ holds too, where $(\tau^{m+1}(f(\psi(q_0))))^+$ is the formula defined above. By the hypothesis of induction, $M' \models \psi$. Thus, $M \models \phi$. //

<u>Theorem 3.2</u> (The main theorem) For any extended formula ϕ of L*, ϕ is internally valid if and only if $f(\phi)$ is valid in $_3L$.

Proof

Let $P_1,\ldots,P_h$ be all predicate symbols appearing in ϕ.

Suppose that ϕ is not internally valid. Then, there exists a model $M=\langle X,u,U\rangle$ such that $M\models\phi$ does not hold. Let $M=\langle X\cup W, v,v\rangle$ be a model such that $v(J_1P_i(x_1,\ldots,x_{i_k},q_0))=u(P_i)$ and $v(J_1P_i(x_1,\ldots,x_{i_k},q_0)\vee J_{1/2}P_i(x_1,\ldots,x_{i_k},q_0))=U(P_i)$ for every $i=1,\ldots,h$, where q_0 is a constant in W. Then, it is obvious that $u(P_i)\subseteq U(P_i)$ for all $i=1,\ldots,h$. Therefore, ${}_3M\models f(\phi)$ does not hold by Lemma 3.1. Hence, $f(\phi)$ is not valid in ${}_3L$.

Conversely, suppose that ${}_3M'\models f(\phi)$ does not hold for some model ${}_3M'=\langle X'\cup W',v',v'\rangle$. Define $M'=\langle X',u',U'\rangle$ by $u'(P_i)=v'(J_1P_i(x_1,\ldots,x_{i_k},q_0))$ and $U'(P_i)=v'(J_1P_i(x_1,\ldots,x_{i_k},q_0)\vee J_{1/2}P_i(x_1,\ldots,x_{i_k},q_0))$ for $i=1,\ldots,h$, where q_0 is a constant in W'. Then, M' is a really incomplete model. Moreover, by Lemma 3.1 $M'\models\phi$ does not hold since ${}_3M\models f(\phi)$ does not hold. Then, ϕ is not internally valid. //

Theorem 3.2 is considered as an embedding theorem of wff's in L^* into ${}_3L$. Therefore, we get the end of this paper. By making use of this theorem, we are able to obtain various interesting theorems. We shall discuss those facts in another paper.

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