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The Asymptotic Behavior of the Free Boundary in a One Phase Stefan Problem

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§1. Introduction

A one phase Stefan problem is a free boundary problem arising from a description of melting ice, where ice is maintained at zero degrees centigrade . In this note we are concerned with a Dirichlet type problem. The situation is roughly described as follows.

Let D be a bounded simply connected domain in R^n, $n \geq 3$, with smooth boundary ∂D . (D is the heat island, so to speak.) The exterior $R^n \setminus \bar{D}$ of D is occupied by water or ice. At $t = 0$ water occupies a domain G bounded by ∂D and another bounded connected smooth hypersurface Γ_0 lying outside D . We suppose that the temperature of water on ∂D for all $t > 0$ is given and positive. Let us denote this boundary value by $g(x,t)$, $t > 0$. Let $W(t)$ be the region occupied by water at time t . As time goes on, $W(t)$ will increase.

In this note we are interested in the following question :

Supposing that $g(x,t)$ satisfies the inequality

$$(1.1) \qquad c_1(1+t)^\beta \leq g(x,t) \leq c_2(1+t)^\beta ,$$

can one prove the estimate of the type

(1.2) $\{ x \mid |x| \leq R_1(1+t)^b\} \subset W(t) \cup D \subset \{x \mid |x| \leq R_2(1+t)^b\}$

and, if so, for what value of b ?

A.Friedman [3],[4] proved (1.2) for the case $\beta = n/2 - 1$ with $b = 1/2$. Recently, S.Tokuda [11] proved the second relation for the case $\beta = 0$ with $b = 1/n$. In this note we extend these results for $0 \leq \beta \leq n/2 - 1$ with $b = (\beta + 1)/n$ (see Theorems 2 and 3). The second relation of (1.2) holds in fact for all $\beta \geq 0$ (Theorem 2) but we do not know if the first relation holds outside the mentioned range of β . Our result will be proved for weak solutions. The proof is based on a simple comparison argument. We compare the solution with a function which may be called a quasi-stationary solution.

§ 2. Formulation of the one phase Stefan problem

2.1. One phase Stefan problem

Let G be a bounded domain in R^n, $n \geq 3$, whose boundary consists of two smooth connected hypersurfaces ∂D and Γ_0 , with $\partial\hat{D}$ lying inside Γ_0 and bounding a simply connected domain D . Further we assume that D contains the origin 0. We put $\Omega = R^n \setminus \bar{D}$. Since $W(t)$, the region occupied by water at time t , increases with t in a one phase problem, we can assume that $W(t)$ can be expressed as

$W(t) = \{(x,t) \in \Omega \times (0,\infty) \mid t > s(x)\}$,

where $s(x)$ is a certain function defined on Ω . The free boundary Γ_t at time t is given by $\Gamma_t = \{x \in \Omega \mid s(x) = t\}$. Now, the Stefan problem we are concerned with is formulated as follows :

Problem (I) Given smooth functions $h(x) \geq 0$, $x \in G$, and $g(x,t) > 0$, $x \in \partial D$, $t > 0$, find (smooth) functions $s(x)$, $x \in \Omega$, and $\Theta(x,t)$, $x \in \Omega$, $t \geq s(x)$, which satisfy

(2.1) $\left(-\Delta + \frac{\partial}{\partial t} \right) \Theta(x,t) = 0 , \quad t > s(x)$,

(2.2) $\Theta(x,0) = h(x) , \quad x \in G$,

(2.3) $\Theta(x,t) = g(x,t) , \quad (x,t) \in \partial D \times (0,\infty)$,

(2.4) $\Theta(x,t) = 0 , \quad t = s(x)$,

(2.5) $\mathrm{grad}_x \Theta \cdot \mathrm{grad}_x s = -\kappa , \quad t = s(x)$,

where κ is a given positive constant.

In several space variables, the existence of a classical solution of Problem (I) is proved only for a small time interval (see E.-I. Hanzawa [6] and A.M. Meǐrmanov[9,10]). In this note we shall prove (1.2) for weak solutions which have been proved to exist. Specifically, we follow the approach of G.Duvaut [2] , A.Friedman and D.Kinderlehrer [5], and formulate the problem as a parabolic variational inequality.

2.2. Variational inequality

According to Duvaut [2], Friedman and Kinderlehrer [5], Problem (I) leads to the variational inequality, expressed in a pointwise form, of the following type. Let $K = \{v \mid v \in H^1(D), v \geq 0\}$.

Problem (II) Given functions $f(x)$, $x \in \Omega$, and $\Psi(x,t) \geq 0$, $x \in \partial D$, $t > 0$, find $u \in L^2_{loc}([0,\infty); H^2_{loc}(\bar{\Omega}))$ such that $u_t \in L^2_{loc}([0,\infty); L^2_{loc}(\bar{\Omega}))$ and

$$\begin{cases} (-\Delta u + u_t)(v-u) \geq f(v-u) & \text{a.e. for } v \in K, \\ u = \Psi & \text{on } \partial D \times (0,\infty), \\ u = 0 & \text{on } D \times \{0\}. \end{cases}$$

Problems (I) and (II) are related as follows. Given g, h, and κ of Problem (I), define f and Ψ as

$$(2.6) \quad \begin{cases} \Psi(x,t) = \int_0^t g(x,\tau)\,d\tau, & x \in \partial D,\ t \geq 0, \\ f(x,t) = \begin{cases} h(x,t) & \text{if } x \in G, \\ -\kappa & \text{if } x \in \Omega \setminus G. \end{cases} \end{cases}$$

It was proved in [5] that Problem (II) has then a unique solution u and that u satisfies $u_t \geq 0$, a.e., and

$$u \in L^\infty_{loc}([0,\infty); H^{2,p}_{loc}(\bar{\Omega})) \quad \text{for } 1 \leq p < \infty,$$

$$u_t \in L^\infty_{loc}([0,\infty); L^\infty_{loc}(\bar{\Omega})) = L^\infty_{loc}(\bar{\Omega} \times [0,\infty)).$$

Thus $u \geq 0$, a.e. . Relate Θ and u through

$$\Theta(x,t) = u_t(x,t),$$

$$(2.7)\quad \begin{cases} u(x,t) = \int_{s(x)}^{t} \Theta(x,\tau)d\tau & \text{if } x \in \Omega\setminus G\ ,\quad s(x) \le t\ , \\ u(x,t) = 0 & \text{if } x \in \Omega\setminus G\ ,\quad 0 \le t \le s\ , \\ u(x,t) = \int_{0}^{t} \Theta(x,\tau)d\tau & \text{if } x \in G\ ,\ 0 \le t\ . \end{cases}$$

Then it was proved in [5] that u is a solution of Problem (II) if and only if Θ is a weak solution of Problem (I) in the sense of [4]. We also remark that u was proved to be continuous in $\Omega \times (0,\infty)$.

The following comparison theorem, also due to [5], will be used as a basic tool in our proof.

<u>Theorem</u> 1. Let u , $\hat{u}$ be solutions of Problem (II) for f , Ψ and $\hat{f}$, $\hat{\Psi}$ respectively. Assume that $f \le \hat{f}$ and $\Psi \le \hat{\Psi}$. Then $u \le \hat{u}$ in Ω .*)

§ 3. Main results

Let G and D be as at the beginning of §2 and let $\Omega = R^n$ Given g, h, and κ as in Problem (II), let u be the solution of Problem (II) with f and Ψ defined as (2.6). Put

(3.1) $W(t) = \{x \in \Omega \mid u(x,t) > 0\ ,\ t > 0\}$.

Our main results can now be formulated as follows.

<u>Theorem</u> 2. If $g(x,t)$ is majorized by a smooth non-decreasing function $\eta(t)$, namely if

(3.2) $0 < g(x,t) \le \eta(t)\ ,\ x \in \partial D\ ,\ t > 0\ ;\ \eta_t \ge 0$

$(\eta_t = \frac{\partial \eta}{\partial t})$, then there exist positive constants a_1

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*) We remark that this theorem applies even if f and Ψ of Problem(II) are not derived from Problem(I).

and a_2 such that

(3.3) $W(t) \subset \{x \in R^n \mid |x|^n < a_1 \int_0^t \eta(\tau) d\tau + a_2\}, \ t > 0$.

<u>Corollary</u> If $\eta(t) = \alpha(1+t)^{\beta}$, $\alpha > 0$, $\beta \geq 0$, we obtain $W(t) \subset \{x \in R^n \mid |x| \leq c_3(1+t)^{(\beta+1)/n}\}, \ t > 0$, where c_3 is a positive constant .

<u>Theorem</u> 3. If there exist constants β , $0 \leq \beta \leq n/2 - 1$, and $k > 0$ such that

(3.4) $k(1+t)^{\beta} \leq g(x,t)$,

then there exist positive constants k' and t_0 such that

(3.5) $W(t) \cup D \supset \{x \in R^n \mid |x| \leq k'(1+t)^{(\beta+1)/n}\}, t > t_0$.

<u>Remark</u> 1. Theorem 3 and Corollary to Theorem 2 show that (1.1) implies (2.1) if $0 \leq \beta \leq n/2 - 1$.

<u>Remark</u> 2. Suppose that D is a ball with center O and $g(x,t) = c(1+t)^{\beta}$. The proof of Theorems 2 and 3 shows that $\partial\Theta/\partial r|_{\partial D}, r = |x|$, is of $O(t^{\beta})$. Thus, (1.2) shows that the amount of heat used to melt ice up to time t is of the same order as the amount of heat having flown in from D up to time t. In fact, the proof shows that the heat retained in water at time t is of lower order if $\beta \leq n/2-1$.

§4. Proof of Theorems 2 and 3

4.1. Upper bound for $W(t)$; the proof of Theorem 2

The proof is based on Theorem 1 . In order to obtain $\tilde{u}$, with which u , the solution of Problem (II), is to be compared, we introduce a melting-ice situation with heat generation in the water region. It is a spherically symmetric problem, and for convenience we express the free boundary as $|x| = \rho(t)$. The function ρ and the temperature function $\Phi(r,t)$, $r = |x|$, are defined as follows.

$$(4.1)\quad \begin{cases} \Phi(r,t) = c(t)\{r^{2-n} - \rho(t)^{2-n}\} , & r\in(0,\rho(t)] , \ t \geq 0 , \\ \rho(t)^n = \dfrac{n(n-2)}{\kappa} \int_0^t c(\tau)\,d\tau + \rho(0)^n , \end{cases}$$

where $c(t)$ is a positive function to be determined later. Note that Φ is determined by $c(t)$ and $\rho(0)$. We use the notation

$$B = \{(x,t) \in (R^n\backslash\{0\})\times R \mid \ |x| \leq \rho(t)\},$$

$$\tilde{s}(x) = \rho^{-1}(x) ,$$

where ρ^{-1} is the inverse function of ρ . The free boundary would then be described by $t = \tilde{s}(x)$.

It is now easy to prove the following properties of Φ .

<u>Lemma</u> 1.

$$\Delta\Phi(|x|,t) = 0 , \ (x,t)\in B , \ t \geq 0 ,$$

$$\Phi(\rho(t),t) = 0 , \ t \geq 0 ,$$

$$\text{grad } \Phi(|x|,t)\cdot\text{grad } \tilde{s}(x) = -\kappa, \ |x| = \rho(t) , \ t \geq 0 .$$

Lemma 2. $c_t(t) \geq 0 \Rightarrow \Phi_t(r,t) \geq 0 \quad , r \leq \rho(t)$.

Proof of Theorem 2. We use function Φ with $c(t)$ equal to $c'\eta(t)$ where c' is a positive constant to be determined. We take $\rho(0)$ large enough so that the set $\{x \,|\, |x| \leq \rho(0)\}$ contains D . It follows from Lemmas 1 and 2 that $(\Phi, \tilde{s})$ is a solution of Problem (I) if we make the following replacement. Firstly, Θ, s and G are replaced by Φ , $\tilde{s}$ and $\tilde{G} = \{x \in \Omega \,|\, |x| < \rho(0)\}$, respectively. Secondly, $h(x)$ and $g(x,t)$ are replaced by the values of $\Phi(|x|,t)$ on the respective set. Finally, equation (2.1) is replaced by an inhomogeneous equation that

$$(-\Delta + \frac{\partial}{\partial t})\Phi(|x|,t) = q(x,t) \quad , \quad t > \tilde{s}(x) \quad ,$$

where

$$(4.2) \quad q(x,t) = \frac{\partial}{\partial t}\Phi(|x|,t) \geq 0 \; .$$

Define now $\tilde{u}$ by (2.7) with obvious replacements. Then, it is seen that $\tilde{u}$ is a solution of Problem (II) with Ψ and f replaced respectively by $\tilde{\Psi}$ and $\tilde{f}$ defined as follows :

$$\tilde{\Psi}(x,t) = \int_0^t \Phi(|x|,\tau)\,d\tau \quad , \quad x \in \partial D \; , \; t > 0 \; ,$$

$$\tilde{f}(x,t) = \begin{cases} \Phi(|x|,0) + Q(x,t) & \text{if } x \in \tilde{G} \; , \; t > 0 \; , \\ -\kappa + Q(x,t) & \text{if } x \in \Omega \setminus \tilde{G} \; , \; t > 0 \; , \end{cases}$$

where we put

$$Q(x,t) = \begin{cases} \int_{\tilde{s}(x)}^{t} q(x,\tau)d\tau , & x \in \Omega \setminus \tilde{G} , t > 0 , \\ \int_0^t q(x,\tau)d\tau , & x \in \tilde{G} , t > 0 . \end{cases}$$

It is clear that by choosing c' large enough and $\rho(0)$ still larger, if necessary, we can make $\tilde{G} \supset G$, $\tilde{\Psi} \geq \Psi$, and $\tilde{f} \geq f$. Applying Theorem 1, we then obtain $\tilde{u} \geq u$. Therefore, u is zero whenever $\tilde{u}$ is zero. Thus, we get

$$\begin{aligned} W(t) &\subset \{x \,|\, |x| < \rho(t)\} \\ &= \{x \,|\, |x|^n < \rho(t)^n\} \\ &= \{x \,|\, |x|^n < c_4 \int_0^t \eta(\tau)d\tau + c_5\} \end{aligned}$$

where $c_4 = \dfrac{n(n-2)c'}{\kappa}$, $c_5 = \rho(0)^n$. Q.E.D.

4.2. Lower bound for W(t) ; the proof of Theorem 3

We first reduce the problem to a spherically symmetric one. Let us denote by B_R the open ball of radius R with the center 0 . Since D is a domain containing 0, we can take R_1 such that $\overline{B_{R_1}} \subset D$. We fix such an R_1 and take R_0 so that $0 < R_0 < R_1$. Letting k and β be constants appearing in (3.4), we introduce a function of the type Φ defined by (4.1). Namely, we put

$$(4.3) \quad \hat{\Phi}(r,t) = kR_0^{n-2}(1+t)^{\beta}\{r^{2-n} - \hat{\rho}(t)^{2-n}\} , \; r \in (0,\hat{\rho}(t)] , t > 0$$

(4.4) $\hat{\rho}(t)^n = \frac{n(n-2)kR_0^{n-2}}{\kappa} \int_0^t (1+\tau)^\beta d\tau + \hat{\rho}(0)^n \ , \ \hat{\rho}(0) = R_1 \ .$

We now consider the following problem :

Problem (III) This is Problem (II) with D , G , etc. replaced by the following $\hat{D}$, $\hat{G}$, etc.

$$\begin{cases} \hat{D} = B_{R_0} \ , \ \hat{G} = B_{R_1} \setminus \overline{B_{R_0}} \ , \\ \hat{h}(x) = \hat{\Phi}(|x|,0) \quad ; \quad \hat{g}(x,t) = \hat{\Phi}(R_0,t) \ , \ |x| = R_0 \ . \end{cases}$$

This problem has a classical solution $\hat{\Theta}(x,t)$ and $\hat{s}(x)$. The pair $(\hat{\Theta}, \hat{s})$ gives rise to a solution $\hat{u}$ of the corresponding variational inequality in $(R^n \setminus \hat{D}) \times (0,\infty)$. Since $R^n \setminus \hat{D} \supset \Omega$, $\hat{u}$ may be regarded as a solution of the variatinal inequality in $\Omega \times (0,\infty)$. Thus, we can compare the solution u of the original problem with $\hat{u}$. Since the boundary value $\hat{g}$ of Problem (III) is non-decreasing and majorizes the initial value, it is clear that $\hat{\Theta}(x,t)\big|_{x \in \partial D} \le \max_{|x|=R_0} \hat{\Theta}(x,t) \le k(1+t)^\beta$, where the last inequality follows from the definition of $\hat{\Phi}$. Therefore, it follows from Theorem 1 that $\hat{u} \le u$. Thus, we have seen that it suffices to prove Theorem 3 for the solution $\hat{\Theta}$(or $\hat{u}$) of Problem (III).

Since $\hat{\Theta}$ and $\hat{u}$ are spherically symmetric, there exists a smooth function $\hat{r}(t)$ such that

$$\hat{W}(t) = \{x \in R^n \setminus \hat{D} \ | \hat{u}(x,t) > 0\} = B_{\hat{r}(t)} \setminus \overline{B_{R_0}} \quad .$$

The following lemma, which expresses the conservative law for heat flow, can be proved easily by integrating the heat equation by parts.

Lemma 3. Denoting by $|A|$ the volume of the set $A \subset R^n$, we have ($\Theta_r = \partial\Theta/\partial r$)

$$(4.5) \quad \int_{\hat{W}(t)} \hat{\Theta}(\xi,t)d\xi - \int_{\hat{G}} \hat{h}(\xi)d\xi + \kappa\{|\hat{W}(t)| - |\hat{G}|\} + \int_0^t d\tau \int_{\partial\hat{D}} \hat{\Theta}_r(\xi,\tau)d\xi = 0$$

On the other hand, we see from the proof of Theorem 2 that

$$(4.6) \quad \hat{\Theta}(x,t) \le \hat{\Phi}(|x|, t), \quad x \in \hat{W}(t), \quad t > 0.$$

Since $\hat{\Theta}$ and $\hat{\Phi}$ take the same boundary values on $\partial\hat{D}$, the following lemma holds. This lemma is crucial, as it gives a lower bound of the amount of heat flowing in through $\partial\hat{D}$.

Lemma 4. We have

$$(4.7) \quad -\hat{\Theta}_r(x,t) \ge -\hat{\Phi}_r(|x|,t), \quad x \in \partial\hat{D}, \quad t > 0.$$

In what follows we denote by $c_1, c_2, \ldots$ various constants which do not depend on t nor R_0. Using (4.7), we see that

$$(4.8) \quad -\int_0^t d\tau \int_{\partial\hat{D}} \hat{\Theta}_r(\xi,\tau)d\xi \ge -\int_0^t d\tau \int_{\partial\hat{D}} \hat{\Phi}_r(|\xi|,\tau)d\xi = c_1 R_0^{n-2} (1+t)^{\beta+1}.$$

Using (4.7), we also obtain

$$(4.9) \quad \int_{\hat{W}(t)} \hat{\Theta}(\xi,t)d\xi \le \int_{\hat{W}(t)} \hat{\Phi}(|\xi|,t)d\xi \le \int_{B_{\hat{\rho}(t)}} \hat{\Phi}(|\xi|,t)d\xi = c_2 R_0^{n-2}(1+t)^{\beta} \int_0^{\hat{\rho}(t)} \{ r - \hat{\rho}(t)^{2-n} r^{n-1}\}dr$$

$$\leq c_3 R_0^{\,n-2}(1+t)^{\beta}\{c_4 R_0^{\,2(n-2)/n}(1+t)^{2(\beta+1)/n} + c_5\}$$

$$= c_6 R_0^{\,n-2+p}(1+t)^{(1+2/n)\beta+2/n} + c_7 R_0^{\,n-2}(1+t)^{\beta} \quad ,$$

where $p = \dfrac{2(n-2)}{n} > 0$.

If $0 \leq \beta < n/2-1$, the right side of (4.9) is of $o((1+t)^{\beta+1})$. Thus, by using (4.8) and (4.9) in (4.5) and recalling that $\hat{h}(x) \geq 0$, we obtain

(4.10) $$\{|\hat{W}(t)| - |\hat{G}|\} \geq c_8 R_0^{\,n-2}(1+t)^{\beta+1} + o((1+t)^{\beta+1}) \ .$$

If $\beta = n/2-1$, the first term on the right side of (4.9) is of order $(1+t)^{\beta+1}$. However, if we take R_o sufficiently small, which is permissible, so that $c_1 - c_6 R_0^{\,p} > 0$, we can obtain (4.10) also for $\beta = n/2-1$. (3.5) for $\hat{W}(t)$ follows from (4.10) at once. Q.E.D.

References

[1]: J.R.Cannon and C.D.Hill, Remarks on a Stefan problem , Jour.Math.Mech. , 17(1967), 433-440 .

[2]: G.Duvaut, Résolution d'un probleme de Stefan (Fusion d'un bloc de glace à zero degreé) , C.R.Acad.Sc.Paris , 276(1973), 1461-1463 .

[3]: A.Friedman, Asymptotic behavior of solutions of parabolic differential equations and of integral equations , in J.Hale and J.La Salle,ed. "Differential Equations and Dynamic Systems", (1967)

[4]: A.Friedman, The Stefan problem in several space variables, Trans.A.M.S. , 133(1968), 51-87 .

[5]: A.Friedman and D.Kinderlehrer, A one phase Stefan problem, Indiana Univ.Math.Jour. , 24(1975), 1005-1035 .

[6]: E.-I. Hanzawa, Classical solutions of the Stefan problem, Tôhoku Math.J., 38(1981), 297-335 .

[7]: H.Kawarada, Stefan-type free boundary problems for heat equations , Publ.RIMS , 9(1974), 517-533 .

[8]: D.Kinderlehrer and L.Nirenberg, Regularity in the free boundary problems , Ann.Scu.Norm.Sup.Pisa , 4(1977), 373-391 .

[9]: A.M. Meǐrmanov, On classical solvability of the multidimensional Stefan problem , Dokl.Akad.Nauk SSSR, 249(1979) = Soviet Math.Dokl. , 20(1979) , 1426-1429 .

[10]: A.M. Meǐrmanov, On the classical solution of the multidimensional Stefan problem for quasilinear parabolic equations , Matem.Sb. , 112(154)(1980), No.2, 170-192 = Math.USSR Sb. , 40(1981), No.2, 157-178 .

[11]: S. Tokuda, On the asymptotic behavior of solutions of the multidimensional one phase Stefan problem , Master's Thesis, Univ. of Tokyo, (1976), (in Japanese) .